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EFFECT OF END CONDITIONS UPON VIBRATIONS
OF THIN WALLED OPEN SECTION BEAMS

by

(C)

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A THESIS

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "EFFECT OF END CONDITIONS UPON VIBRATIONS OF THIN WALLED OPEN SECTION BEAMS" submitted by GEORGE EDSEL GILLESPIE in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The effect of elastic boundary conditions upon the coupled bending vibrations and torsional vibration of a thin walled beam of open cross section with an applied axial load is investigated. Results are presented for elastic end conditions and also for fixed supports and simple supports.

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NOTATION

b	total length of middle line of cross section
c	torsion constant
c_w	warping constant
c_y, c_z	coordinates of centroide
i	integer to indicate a given term in a series
λ	length of the beam
m	mass density of the material in the beam
n	integer, indicative of deflection mode
p_n	natural frequency of vibration
r	roots of characteristic equation
r_t	perpendicular distance from the tangent at the point under consideration on the midline to the axis of rotation
s	distance along the midline to the point under consideration
t	time, thickness
u	warping displacement of a cross section
v, w	deflection of the shear center in the y and z directions respectively
w_s, \bar{w}_s	warping function, mean value of warping function
x, y, z	Cartesian coordinates
A	area of the cross section
D	operator $D = \frac{d}{dx}$
E	Young's modulus of elasticity

G	modulus of rigidity
I_0, I_p	polar moments of inertia about shear center, and centroide respectively
I_η, I_ζ	principal centroidal moments of inertia
M_t	torque
M_y, M_z	moments about y and z axes of the cross section
P	axial thrust
$T(t)$	time function
V	shear in the member
$V(x), W(x)$	deflection of shear center in y and z directions, respectively, as functions of distance along the beam only
w_t	intensity of distributed torque acting along the shear center axis
w_y, w_z	lateral loads in the y and z directions respectively
γ	Function $\gamma_i, \gamma_{i+1} = -\frac{\frac{EI_\zeta}{mA} r_i^4 + \frac{Pr_i^2}{mA} - p_n^2}{\frac{EI_\eta}{mA} r_i^4 + \frac{Pr_i^2}{mA} - p_n^2}$
ϵ	strain
θ	angle of twist per unit length
ξ, η, ζ	Cartesian coordinates
ρ_y, ρ_z	radii of curvature of the beam
σ	normal stress
τ	shear stress
ϕ	angle of twist

x

$$\text{Function } x_i, x_{i+1} = \frac{\left[\frac{EI_n}{m\bar{A}} r_i^4 + \frac{P}{m\bar{A}} r_i^2 - p_n^2 \right]}{c_y \left[\frac{P}{m\bar{A}} r_i^2 - p_n^2 \right]}$$

\psi

$$\text{Function } \psi_i, \psi_{i+1} = \frac{\left[\frac{EI_\zeta}{m\bar{A}} r_i^4 + \frac{P}{m\bar{A}} r_i^2 - p_n^2 \right]}{c_z \left[\frac{P}{m\bar{A}} r_i^2 - p_n^2 \right]}$$

\Phi(x)

angle of rotation of cross section as function of x only

CHAPTER I

INTRODUCTION

1.1 PURPOSE OF THESIS

This thesis reports on the investigation of a proposed experimental method for determining the axial load in a member of an existing structure. This method is of practical importance for verifying the theoretically calculated forces in axially loaded members of complicated structures and for measuring the change of load in members of structures which are settling. Plots of the three lowest theoretical natural frequencies versus axial load are prepared, with given end conditions as a parameter. The axial load in a single span member may then be determined by comparing its experimentally measured fundamental frequency of vibration with the corresponding theoretical plot for a member with the same cross section and support conditions.

This necessitates a detailed investigation into the effect of end conditions upon the vibrations of a member of arbitrary cross section under various applied axial loads, which is the major topic of the thesis.

1.2 HISTORICAL REVIEW

The vibration of a thin walled open cross sectioned member under an eccentrically applied axial load were discussed by V.Z. Vlasov (1,2) in 1940. He derived the differential equations for coupled vibrations and solved them for the case of simple supports which in this thesis means that

the member is not allowed to deflect perpendicular to the longitudinal axis or to rotate about the longitudinal axis. Vlasov's solution employed the method of separation of variables and assumed the deflected form to be a sine function. Timoshenko (3) considered the problem of finding the natural frequencies for the coupled torsional and bending vibrations for a channel cross section having simple supports. In 1940 Garland (4) used the method of Rayleigh-Ritz to determine the frequencies and amplitude ratios of a cantilever beam with channel cross section. He verified his results by experiment. Frederhoffer (5) solved the general case for a symmetrical cross section. A solution for an arbitrary cross section using energy methods was presented by Karyakin (6).

In 1954 Gere (7) analysed the free torsional vibrations of bars of thin walled open cross sections for which the shear center and the centroid coincide. He investigated the effect of warping on the frequency of single span members with various end conditions. Gere and Lin (8) solved the differential equations for coupled free vibrations of beams of non symmetric open cross section. They considered three different end conditions: simple supports, fixed ends and a cantilever. They solved their derived equations by an exact method, using a computer to do the calculations. They also used the Rayleigh-Ritz method to obtain an approximate solution.

Aggarwall and Cranch (9) further extended the theory of coupled bending torsional vibrations by including the effects of rotatory inertia and transverse shear deformation. Their results agree with the Timoshenko theory for the low frequencies and give more satisfactory results for high

frequencies since it predicts finite velocities of wave propagation for high frequencies whereas the Timoshenko theory predicts infinite velocities.

Didrikson (10) applied the computer method of Gere and Lin to the problem of thin walled members of open cross section with an axial load applied at the centroid of the cross section. Didrikson's experimentally determined frequencies were less than those calculated assuming fixed ends but greater than those calculated assuming simple supports. This indicated that the actual support conditions were between the fixed and simple cases, and that the theory should be modified to account for elastic end conditions. Rogers (11) discussed elastic end conditions for bending vibrations.

1.3 OUTLINE OF INVESTIGATION

The vibrations considered are torsional as well as flexural since the member was of unsymmetric thin walled open cross section. Torsion of a beam generally produces warping of the cross section. Warping is the movement in the longitudinal direction of a point out of the plane perpendicular to the longitudinal axis of the member. If a member has torque applied only at the ends and all cross sections are free to warp then the member is said to be in pure torsion. A member is under nonuniform torsion if the torque varies along the member or if some of the cross sections are restrained from warping. In nonuniform torsion the warping varies along the length of the member so that axial stress accompanies the torsional shear stress.

The torsional and flexural vibrations are coupled when the axis joining the shear center of the cross sections of the member does not coin-

cide with the longitudinal axis. The axis through the shear center is the only axis of the member where an applied transverse load will produce flexural deformations only. Since the flexural and torsional frequencies are influenced by the end supports of the member, support conditions for a vibrating member are discussed in detail in the following chapter.

CHAPTER II

THEORY

This chapter outlines the derivation of the differential equations describing the free coupled bending torsional vibrations of a thin walled, open cross section member under a concentric axial load. These differential equations are solved for the lower natural modes of vibration employing the method used by Gere and Lin (8) when they solved the problem for zero axial load. The boundary conditions considered in this solution are elastic. Ordinary differential equations are obtained from the partial differential equations by the method of separation of variables. Rotatory inertia and shear deformation are neglected since only the lower fundamental frequencies are investigated.

In the derivation of the differential equations it is assumed that the member has thin walls, is long and slender with a constant cross section, and is composed of a homogeneous material. It is further assumed that the angle of rotation and the transverse deflections are small. Bending is taken about the centroidal axes and twisting about the longitudinal axis through the shear center. The thrust is assumed to act along a line parallel to the longitudinal axis of the member.

2.1 DERIVATION OF GOVERNING EQUATIONS

The coordinate system used in the derivation of the equations is shown in figures 2.1 and 2.2. The y and z axes have their origin at

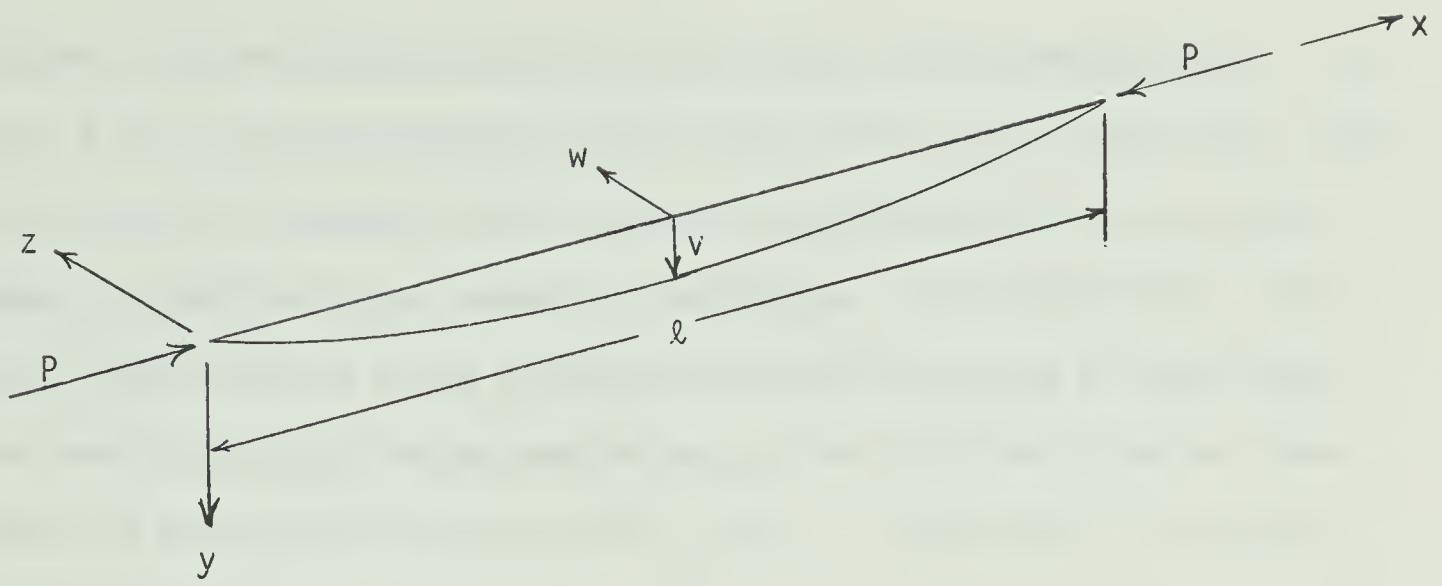


FIGURE 2.1 COORDINATE SYSTEM

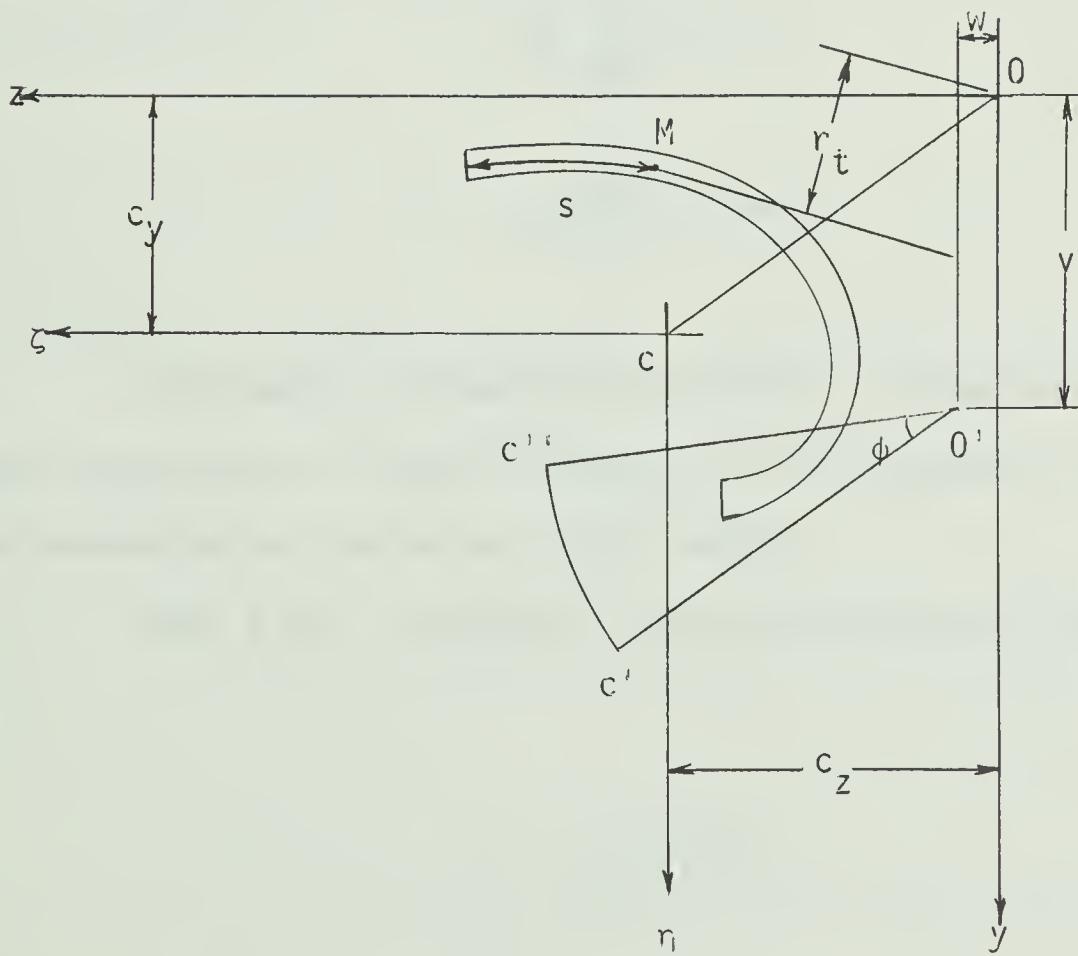


Figure 2.2 ORIENTATION OF PRINCIPAL AXES

the shear center and are parallel to the η and ζ axes respectively. The η and ζ axes are the principal axes of the cross section with their origin at the center of gravity. The x axis is the longitudinal axis of the member through the shear center of the section. The deflection in the y and z directions are v and w respectively with the angle of twist about the longitudinal axis being denoted as ϕ . The distances from the shear center to the center of gravity are c_y and c_z in the y and z directions respectively.

The bending of the member about the z and y axes by a transverse load is expressed by

$$EI_{\zeta} \frac{d^2v}{dx^2} = -M_z \quad 2.1$$

$$EI_{\eta} \frac{d^2w}{dx^2} = -M_y \quad 2.2$$

The theory of nonuniform torsion of a member with a thin walled open cross section is here discussed as it is necessary for derivation of the expression for torque on a cross section.

When a bar is subjected to pure torsion the torque M_t is given by

$$M_t = Gc\theta \quad 2.3$$

where G is the torsional rigidity of the bar

θ is the constant angle of twist per unit length

c is the torsion constant

The value of the torsion constant for a cross section which consists of n thin bars of different thickness is given by

$$c = \frac{1}{3} \sum_{i=1}^n m_i t_i^3$$

where m_i is the length of the middle line of the ith member

t_i is the thickness of the ith member

The warping displacement u for pure torsion is given by

Timoshenko (3) as

$$u = \theta (\bar{w}_s - w_s)$$

The quantity w_s is called the warping function and is defined by

$$w_s = \int_0^s r_t ds$$

where r_t is the perpendicular distance from the tangent at a point M on the midline of the member cross section to the axis of rotation as shown in figure 2.2,

s is the distance along the middle of the member to the point M under consideration.

The quantity \bar{w}_s is the average value of w_s and is therefore defined as

$$\bar{w}_s = \frac{1}{b} \int_0^b w_s \, ds$$

where b is the total length of the midline of the cross section.

For nonuniform torsion the constant angle of twist, θ , is replaced by the rate of change of rotation about the longitudinal axis, $\frac{d\phi}{dx}$, in the formula for warping displacement.

$$u = \frac{d\phi}{dx} (\bar{w}_s - w_s) \quad 2.4$$

The strain due to the warping displacement u is

$$\varepsilon_x = \frac{du}{dx} = \frac{d^2\phi}{dx^2} (\bar{w}_s - w_s) \quad 2.5$$

Assuming the material obeys Hooke's Law the longitudinal stress due to warping is

$$\sigma_x = E \frac{d^2\phi}{dx^2} (\bar{w}_s - w_s) \quad 2.6$$

This longitudinal stress produces a shearing stress τ which produces a torque in the cross section. The shearing stress is given by Timoshenko (3) to be

$$\tau = - \frac{E}{t} \frac{d^3\phi}{dx^3} \int_0^s (\bar{w}_s - w_s) t \, ds$$

Thus the torque due to warping of the cross section is

$$\begin{aligned}
 M_{t2} &= \int_0^b \tau t r_t ds \\
 &= -Ec_w \frac{d^3\phi}{dx^3}
 \end{aligned} \tag{2.7}$$

where c_w is the warping function of the cross section, and is given by

$$c_w = \int_0^b (\bar{w}_s - w_s)^2 t ds$$

The total torque acting on a bar subjected to nonuniform torsion is obtained by adding equation 2.3 to equation 2.7

$$M_t = Gc \frac{d\phi}{dx} - Ec_w \frac{d^3\phi}{dx^3} \tag{2.8}$$

Differentiating equation 2.8 with respect to x gives the equation of non-uniform torsion under static torque

$$-w_t = Gc \frac{d^2\phi}{dx^2} - Ec_w \frac{d^4\phi}{dx^4} \tag{2.9}$$

A member of thin walled open cross section under arbitrary lateral loading deforms according to equations 2.1, 2.2, and 2.8.

The effect of an axial load upon the bending and torsion equation is now shown. Consider the point M on the mid-surface of the member cross section as shown in figure 2.3. After the axial load is applied the member deforms with the point M moving v_s and w_s in the y and z directions respectively

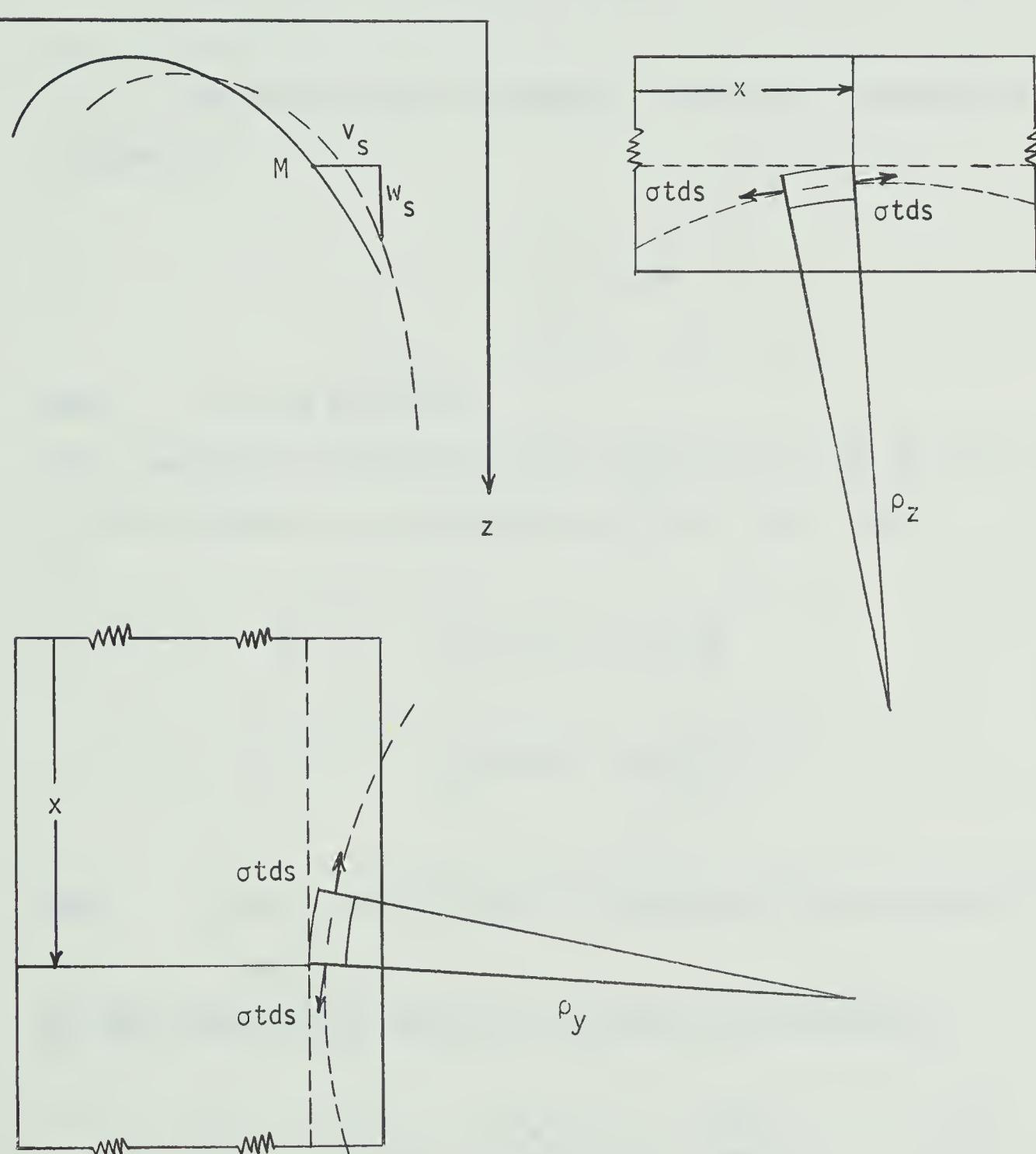


Figure 2.3 MEMBER OF ARBITRARY UNSYMMETRICAL OPEN
CROSS SECTION SUBJECTED TO AXIAL THRUST

$$v_s = v - \phi z$$

$$w_s = w + \phi y$$

The axial load on the member is positive in compression and is given by

$$P = \int_0^b \sigma t \, ds$$

where t is the thickness.

Let p_z denote the difference of the projection of $\sigma t \, ds$ on the z axis and p_y the difference of the projection upon the y axis. Thus

$$p_z \, dx \, ds = \sigma t \, ds \frac{dx}{\rho_z}$$

2.10

$$p_y \, ds \, dx = \sigma t \, ds \frac{dx}{\rho_y}$$

where ρ_z and ρ_y are the radii of curvature of the proejction of $v_s(x)$ and $w_s(x)$.

For small deflections the radii of curvature are defined by

$$\frac{1}{\rho_z} = - \frac{d^2 w_s}{dx^2}, \quad \frac{1}{\rho_y} = - \frac{d^2 v_s}{dx^2}$$

Substituting for the curvature and dividing by $dx \, ds$ gives

$$p_z = - \sigma t \left[\frac{d^2 w}{dx^2} + y \frac{d^2 \phi}{dx^2} \right]$$

$$p_y = -\sigma t \left[\frac{d^2 v}{dx^2} - z \frac{d^2 \phi}{dx^2} \right]$$

These loads give rise to a torque dW_t about the x axis. The value of the torque is the load p_y times its moment arm from the x axis and p_z times its moment arm. Therefore

$$dW_t = p_z y \, ds - p_y z \, ds$$

$$\begin{aligned} dW_t &= -\sigma t \left[\frac{d^2 w}{dx^2} + y \frac{d^2 \phi}{dx^2} \right] y \, ds \\ &\quad + \sigma t \left[\frac{d^2 v}{dx^2} - z \frac{d^2 \phi}{dx^2} \right] z \, ds \end{aligned}$$

Integrating dW_t over the cross section and noting that

$$\int_A y \, t \, ds = A c_y \quad \int_A z \, t \, ds = A c_z \quad \int_A zy \, t \, ds = 0$$

$$\int_A y^2 \, t \, ds = I_z \quad \int_A z^2 \, t \, ds = I_y$$

the total torque $(W_t)_p$ due to the axial load is obtained

$$(W_t)_p = \sigma \frac{d^2 v}{dx^2} c_z A - \sigma I_\eta \frac{d^2 \phi}{dx^2} - \sigma \frac{d^2 w}{dx^2} c_y A - \sigma I_\zeta \frac{d^2 \phi}{dx^2}$$

$$(W_t)_p = \frac{d^2 v}{dx^2} P c_z - \frac{d^2 w}{dx^2} P c_y - \frac{\rho I_0}{A} \frac{d^2 \phi}{dx^2} \quad 2.11$$

The bending moments caused by the axial load p are

$$(M_\zeta)_p = P(v + c_z \phi) \quad 2.12$$

$$(M_\eta)_p = P(w - c_y \phi) \quad 2.13$$

The bending equations for a member with an arbitrary lateral load and an axial thrust are obtained by combining equation 2.12 with 2.1 and 2.13 with 2.2. The resulting equations are

$$EI_\zeta \frac{d^2v}{dx^2} = -M_\zeta - P(v - c_z \phi)$$

$$EI_\eta \frac{d^2w}{dx^2} = -M_\eta - P(w + c_y \phi)$$

These equations are expressed in terms of the loads in the y and z directions by differentiating twice with respect to x to obtain

$$EI_\zeta \frac{d^4v}{dx^4} = w_y - P \frac{d^2v}{dx^2} + c_z \frac{d^2\phi}{dx^2} \quad 2.14$$

$$EI_\eta \frac{d^4w}{dx^4} = w_z - P \frac{d^2w}{dx^2} - c_y \frac{d^2\phi}{dx^2} \quad 2.15$$

The torque equation obtained by combining equation 2.11 with 2.9 is

$$-W_t = -Ec_w \frac{d^4\phi}{dx^4} + (Gc - \frac{P_{I_0}}{A}) \frac{d^2\phi}{dx^2} - P_{C_y} \frac{d^2w}{dx^2} + P_{C_z} \frac{d^2v}{dx^2} \quad 2.16$$

For free vibrations the only loads acting on the member other than the axial load are the loads due to inertia forces. Therefore the components of the loading in the transverse direction are

$$W_y = -mA \frac{\partial^2}{\partial t^2} (v - c_z \phi)$$

$$W_z = -mA \frac{\partial^2}{\partial t^2} (w + c_y \phi)$$

The inertial torque is

$$\begin{aligned} (W_t)_I &= -I_p \frac{\partial^2\phi}{\partial t^2} + mA \frac{\partial^2}{\partial t^2} (v - \phi c_z) c_z - mA \frac{\partial^2}{\partial t^2} (w + c_y \phi) c_y \\ &= -mI_0 \frac{\partial^2}{\partial t^2} + mA (c_z \frac{\partial^2 v}{\partial t^2} - c_y \frac{\partial^2 w}{\partial t^2}) \end{aligned}$$

When the inertial loads are substituted into equations 2.14, 2.15 and 2.16 the differential equations describing the free vibrations of a thin walled member of open cross section under an applied axial load are obtained. They are

$$EI \zeta \frac{\partial^4 v}{\partial x^4} + P \frac{\partial^2 v}{\partial x^2} + mA \frac{\partial^2 v}{\partial t^2} - P_{C_z} \frac{\partial^2 \phi}{\partial x^2} - mAc_z \frac{\partial^2 \phi}{\partial t^2} = 0 \quad 2.17$$

$$EI \eta \frac{\partial^4 w}{\partial x^4} + P \frac{\partial^2 w}{\partial x^2} + mA \frac{\partial^2 w}{\partial t^2} + PC_y \frac{\partial^2 \phi}{\partial x^2} + mAc_y \frac{\partial^2 \phi}{\partial t^2} = 0 \quad 2.18$$

$$Ec_w \frac{\partial^4 \phi}{\partial x^4} + [\frac{PI_0}{A} - Gc] \frac{\partial^2 \phi}{\partial x^2} + mI_0 \frac{\partial^2 \phi}{\partial t^2} - PC_z \frac{\partial^2 v}{\partial x^2} - mAc_z \frac{\partial^2 v}{\partial t^2} \\ + PC_y \frac{\partial^2 w}{\partial x^2} + mAc_y \frac{\partial^2 w}{\partial t^2} = 0 \quad 2.19$$

The above equations do not consider the effects of rotatory inertia or shear deflection. Both effects were considered by Aggarwal and Cranch (9) for a member with zero axial load. They stated that rotatory inertia and shear deflections have a large effect only upon the high vibrational frequencies. In this thesis only the three lowest natural frequencies are considered so the terms accounting for rotatory inertia and shear deflections are neglected. The equations derived are similar to those given by Vlasov (2).

2.2 BOUNDARY CONDITIONS

Before considering the case of elastic boundary conditions, the limiting cases of simple supports, fixed supports and free ends will be discussed.

Simple supports do not allow any deflection of the member at the supports, but they do not offer any resistance to rotation of the member about an axis perpendicular to the longitudinal axis. A simple support provides restraint against rotation about the longitudinal axis. Warping of the cross section is not restrained, which makes the stress in the longitudinal direction due to warping equal to zero. Considering the above de-

definition of simple supports and equation 2.6 for the stress due to restrained warping, the boundary conditions for simple supports are seen to be

$$v = w = \phi = 0$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

A fixed support is one which clamps the member rigidly. There is complete resistance to translation of the member and to any rotation about an axis perpendicular to the longitudinal axis of the member. A fixed support does not allow any rotation about the longitudinal axis nor any warping of the cross section. Equation 2.4 is zero when the warping displacements are zero. Thus the boundary conditions for a fixed support are

$$v = w = \phi = 0$$

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} = 0$$

When the support condition at one end of the member is free there is no resistance to translation or rotation. This means that the shear at the end and the bending moment at the end must be zero. Also, the longitudinal stress due to warping and the total torque must be zero. From equations 2.6 and 2.8, and the above requirements the boundary condi-

tions for a free end are

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\frac{\partial^3 v}{\partial x^3} = \frac{\partial^3 w}{\partial x^3} = 0$$

$$Gc \frac{\partial \phi}{\partial x} - Ec_w \frac{\partial^3 \phi}{\partial x^3} = 0$$

A member on elastic supports may be considered to be on a set of supporting springs. One spring resists transverse deflections of the member and another resists rotation. The force in the spring resisting deflections must be equal to the shearing force at the end of the member, and the moment supplied by the spring resisting rotation of the end must equal the bending moment at that end. A third spring resists the displacements due to warping and a fourth resists rotation about the longitudinal axis of the member. Therefore, the force applied by the third spring equals the resultant of the longitudinal stress in the beam due to warping of the cross section, and the torque in the fourth spring equals the torque at the end of the beam. These conditions are expressed mathematically as

$$\bar{V} = k_1 \bar{u}$$

$$\bar{M} = \beta_1 \bar{\theta}$$

$$\frac{\partial^2 \phi}{\partial x^2} (\bar{w}_s - w_s) = k_2 \frac{\partial \phi}{\partial x} (\bar{w}_s - w_s)$$

$$Gc \frac{\partial \phi}{\partial x} + Ec_w \frac{\partial^3 \phi}{\partial x^3} = \beta_2 \phi$$

where \bar{u} is the vectoral displacement of the member

$\bar{\theta}$ is the angle of rotation of the member about an axis
perpendicular to the longitudinal axis

k_1 , k_2 , β_1 , and β_2 are the spring constants

\bar{u} and $\bar{\theta}$ can be broken into components parallel to the given axes of the member. When this is done the boundary conditions become

$$EI_\zeta \frac{\partial^3 v}{\partial x^3} + k_1 v = 0$$

$$EI_\eta \frac{\partial^3 w}{\partial x^3} + k_1 w = 0$$

$$EI_\zeta \frac{\partial^2 v}{\partial x^2} + \beta_1 \frac{\partial v}{\partial x} = 0$$

$$EI_\eta \frac{\partial^2 w}{\partial x^2} + \beta_1 \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + k_2 \frac{\partial \phi}{\partial x} = 0$$

$$Gc \frac{\partial \phi}{\partial x} + Ec_w \frac{\partial^3 \phi}{\partial x^3} - \beta_2 \phi = 0$$

These equations incorporate the three boundary conditions pre-

viously discussed as special cases. For the case of simple supports the values of the spring constants k_1 , β_1 , k_2 , and β_2 are infinity, zero, zero and infinity respectively. For fixed ends the values of the supporting spring constants are all equal to infinity, and for free ends the spring constant values are all zero.

2.3 SEPARATION OF VARIABLES

The governing partial differential equations 2.17, 2.18, and 2.19 can be reduced to a set of ordinary differential equations by the application of the method of separation of variables. Let

$$v(x, t) = V(x) T(t)$$

$$w(x, t) = W(x) T(t)$$

$$\phi(x, t) = \Phi(x) T(t)$$

When these expressions are substituted into the equations they yield the following ordinary differential equations

$$\frac{d^2 T}{dt^2} + p_n^2 T = 0 \quad 2.20$$

$$EI_z \frac{d^4 V}{dx^4} + p \frac{d^2 V}{dx^2} - mAp_n^2 V - P c_z \frac{d^2 \Phi}{dx^2} + mAp_n^2 c_z \Phi = 0 \quad 2.21$$

$$EI_y \frac{d^4 W}{dx^4} + p \frac{d^2 W}{dx^2} - mAp_n^2 W + P c_y \frac{d^2 \Phi}{dx^2} - mAp_n^2 c_y \Phi = 0 \quad 2.22$$

$$\begin{aligned}
 Ec_w \frac{d^4 \Phi}{dx^4} + \left(\frac{PI_0}{A} - Gc \right) \frac{d^2 \Phi}{dx^2} - mI_0 p_n^2 \Phi - P_{cz} \frac{d^2 V}{dx^2} \\
 + mAc_z p_n^2 V + P_{cy} \frac{d^2 W}{dx^2} - mAp_n^2 c_y W = 0
 \end{aligned} \quad 2.23$$

where p_n^2 is the constant of separation.

The solution of the time equation 2.20 is

$$T = A \cos p_m t + B \sin p_m t$$

This shows that the constant of separation p_n^2 is the square of the circular frequency of vibration.

2.4 SOLUTION OF THE EQUATIONS WITH ELASTIC BOUNDARY CONDITIONS

The method of solution of the equations will be outlined for elastic boundary conditions. This method can be applied to the three limiting cases by the proper choice of the supporting spring constants.

The ordinary differential equations 2.21, 2.22, and 2.23 can be rewritten using the operator $D = \frac{d}{dx}$ in the form

$$(EI_\zeta D^4 + PD^2 - mAp_n^2)V + (P_{cz} D^2 + mAp_n^2 c_z)\Phi = 0 \quad 2.24$$

$$(EI_n D^4 + PD^2 - mAp_n^2)W + (P_{cy} D^2 - mAp_n^2 c_y)\Phi = 0 \quad 2.25$$

$$\begin{aligned}
 [Ec_w D^4 + \left(\frac{PI_0}{A} - Gc \right) D^4 - mI_0 p_n^2] \Phi - [P_{cz} D^4 + mAc_z p_n^2] V \\
 + [P_{cy} D^2 - mAp_n^2 c_y] W = 0
 \end{aligned} \quad 2.26$$

W and Φ are eliminated from these equations to obtain the following twelfth order differential equation

$$\begin{aligned}
 & \frac{E^3 c_w I \zeta I_n}{m^3 A^2 I_0} D^{12} V + \frac{E^2}{m^3 A^2} \left\{ P \frac{c_w I_p}{I_0} + \frac{I \zeta I_n}{A} \right\} - \frac{Gc I \zeta I_n}{I_0} D^{10} V \\
 & \left\{ - \frac{p_n^2 E^2}{m^2 A^2} \left[\frac{c_w I_p}{I_0} + \frac{I \zeta I_n}{A} \right] + \frac{EP}{m^3 A^2} \left[P \left[\frac{c_w}{I_0} + \frac{I_p}{A} \right] - \frac{1}{I_0} (c_z^2 I_n + c_y^2 I_\zeta) \right] \right. \\
 & \quad \left. - \frac{Gc I_p}{I_0} \right\} D^8 V + \left\{ - \frac{p_n^2 E}{m^2 A} \left[2P \left[\frac{c_w}{I_0} + \frac{I_p}{A} \right] - \frac{1}{I_0} (c_z^2 I_n + c_y^2 I_\zeta) \right] \right. \\
 & \quad \left. - \frac{Gc I_p}{I_0} \right\} + \frac{P^2}{m^3 A^2} \left[\frac{PI_p}{I_0} - \frac{Gc}{I_0} \right] D^6 V + \left\{ - \frac{p_n^2 p}{m^2 A} \left[\frac{3PI_p}{I_0 A} - \frac{2Gc}{I_0} \right] \right. \\
 & \quad \left. + \frac{p_n^4 E}{m} \left[\frac{c_w}{I_0} + \frac{I_p}{A} - \frac{1}{I_0} (c_z^2 I_n + c_y^2 I_\zeta) \right] \right\} D^4 V \\
 & \quad + \frac{p_n^4}{m} \left[\frac{3PI_p}{I_0 A} - \frac{Gc}{I_0} \right] D^2 V - p_n^6 \frac{I_p}{I_0} = 0
 \end{aligned} \tag{2.27}$$

The system of equations can be used to produce the same equation with W or Φ interchanged for V . Solutions to the twelfth order differential equations can be expressed in the following forms

$$V = \sum_{m=1}^{12} A_m e^{r_m x}$$

$$W = \sum_{m=1}^{12} B_m e^{r_m x}$$

$$\Phi = \sum_{m=1}^{12} C_m e^{r_m x}$$

The characteristic equation of the problem is derived by substi-

tuting the solution for V into equation 2.27. It is

$$\begin{aligned}
 & \frac{E^3 c_w I \zeta I_n}{m^3 A^2 I_0} r^{12} + \frac{E^2}{m^3 A^2} \left\{ P \left[\frac{c_w I_p}{I_0} + \frac{I \zeta I_n}{A} \right] - \frac{Gc I \zeta I_n}{I_0} \right\} r^{10} \\
 & + \left\{ - \frac{p_n^2 E^2}{m^2 A^2} \left[\frac{c_w I_p}{I_0} + \frac{I \zeta I_n}{A} \right] + \frac{EP}{m^3 A^2} \left[P \left[\frac{c_w}{I_0} + \frac{I_p}{A} \right] - \frac{1}{I_0} (c_z^2 I_n + c_y^2 I_\zeta) \right] \right. \\
 & \left. - \frac{Gc I_p}{I_0} \right\} r^8 + \left\{ \frac{P^2}{m^3 A^2} \left[\frac{PI_p}{AI_0} - \frac{Gc}{I_0} \right] - \frac{p_n^2 E}{m^2 A} \left[2P \left[\frac{c_w}{I_0} + \frac{I_p}{A} \right] - \frac{1}{I_0} (c_z^2 I_n + c_y^2 I_\zeta) \right] \right. \\
 & \left. - \frac{Gc I_p}{I_0} \right\} r^6 + \left\{ \frac{p_n^4 E}{m} \left[\frac{c_w}{I_0} + \frac{I_p}{A} - \frac{1}{I_0} (c_z^2 I_n + c_y^2 I_\zeta) \right] \right. \\
 & \left. - \frac{p_n^2}{m^2 A} \left[\frac{3PI_p}{I_0 A} - \frac{2Gc}{I_0} \right] \right\} r^4 + \frac{p_n^4}{m} \left[\frac{3PI_p}{I_0} - \frac{Gc}{I_0} \right] r^2 - p_n^6 \frac{I_p}{I_0} = 0 \quad 2.28
 \end{aligned}$$

If the square of the frequency p_n^2 is known then the equation can be solved for the roots which will be in positive and negative pairs since only even powers of r appear. Therefore, the solution may be written as

$$V = A_1 e^{r_1 x} + A_2 e^{-r_1 x} + A_3 e^{r_2 x} + A_4 e^{-r_2 x} + \dots + A_{12} e^{-r_6 x}$$

$$W = B_1 e^{r_1 x} + B_2 e^{-r_1 x} + \dots + B_{12} e^{-r_6 x}$$

$$\Phi = C_1 e^{r_1 x} + C_2 e^{-r_1 x} + \dots + C_{12} e^{-r_6 x}$$

The arbitrary constants B_i and C_i can be written in terms of the

constants A_i by substituting the solution for V , W , and Φ into equations 2.24 and 2.25 to obtain the following equations

Since $e^{r_1 x}$, $e^{-r_1 x}$, $e^{r_2 x}$, ..., $e^{-r_6 x}$ are linearly independent the coefficients of each must be zero. This enables the constants C_i and B_i to be expressed in terms of A_i . Let

$$\psi_1 = \frac{c_1}{A_1} = \frac{c_2}{A_2} = \frac{\frac{E}{mA} I_s r_1^4}{c_z \left(\frac{P}{mA} r_1^2 - p_n^2 \right)}$$

$$\psi_2 = \frac{C_3}{A_3} = \frac{C_4}{A_4} = \frac{\frac{EI}{mA} r_2^4 + \frac{P}{mA} r_2^2 - p_n^2}{c_z \left(\frac{P}{mA} r_2^2 - p_n^2 \right)}$$

$$x_1 = \frac{B_1}{C_1} = \frac{B_2}{C_2} = - \frac{c_y \left(\frac{Pr_1^2}{mA} - p_n^2 \right)}{\frac{EI}{mA} r_1^4 + \frac{P}{mA} r_1^2 - p_n^2}$$

$$x_2 = \frac{B_3}{C_3} = \frac{B_4}{C_4} = - \frac{c_y \left(\frac{Pr_2^2}{mA} - p_n^2 \right)}{\frac{EI}{mA} r_2^4 + \frac{P}{mA} r_2^2 - p_n^2}$$

Put $\gamma_1 = \psi_1 x_1 = \frac{C_1}{A_1} \frac{B_1}{C_1} = \frac{C_2}{A_2} \frac{B_2}{C_2} = - \frac{\frac{EI}{mA} r_1^4 + \frac{Pr_1^2}{mA} - p_n^2}{\frac{EI}{mA} r_1^4 + \frac{Pr_1^2}{mA} - p_n^2}$

$$\gamma_2 = \psi_2 x_2 = \frac{C_3}{A_3} \frac{B_3}{C_3} = \frac{C_4}{A_4} \frac{B_4}{C_4} = - \frac{\frac{EI}{mA} r_2^4 + \frac{Pr_2^2}{mA} - p_n^2}{\frac{EI}{mA} r_2^4 + \frac{Pr_2^2}{mA} - p_n^2}$$

Using the ratios γ_i and ψ_i the coefficients B_i and C_i can be written in terms of the coefficients A_i since

$$B_i = \gamma_i A_i$$

$$C_i = \psi_i A_i$$

The values of the roots r^2 will not all be positive; therefore,

some of the exponents will be imaginary and the exponentials may be combined to form sines and cosines. The solution will take the following form, assuming the second root is negative

$$V = A_1 e^{r_1 x} + A_2 e^{-r_1 x} + A_3^1 \cos r_2 x + A_4^1 \sin r_2 x + \dots$$

$$W = \gamma_1 A_1 e^{r_1 x} + \gamma_1 A_2 e^{-r_1 x} + \gamma_2 A_3^1 \cos r_2 x + \gamma_2 A_4^1 \sin r_2 x + \dots \quad 2.29$$

$$\Phi = \psi_1 A_1 e^{r_1 x} + \psi_1 A_2 e^{-r_1 x} + \psi_2 A_3^1 \cos r_2 x + \psi_2 A_4^1 \sin r_2 x + \dots$$

where $A_3^1 = A_3 + A_4$

$$A_4^1 = \sqrt{-1} (A_3 - A_4)$$

The unknowns of the problem as stated are p_n with a corresponding twelve values of r , and for each pair p_n and r these are twelve constants A_i . There are thirteen equations derived to solve for the unknowns. These equations are the boundary equations at each end of the member and the characteristic equation 2.28. If a complete solution to the problem were desired then it could be found in terms of one of the constants A_i , which in turn could be found by the use of a given set of initial conditions and equation 2.20. Of main interest in this thesis is a solution for the natural frequencies of the equation which do not depend upon the initial conditions of the problem, but do depend upon the boundary conditions. The following symmetric elastic boundary conditions are considered

$$EI_{\zeta} \frac{d^3V}{dx^3} + k_1 V = 0 \quad \text{at } x = 0 \text{ and } x = \ell$$

$$EI_{\eta} \frac{d^3W}{dx^3} + k_1 W = 0 \quad \text{at } x = 0 \text{ and } x = \ell$$

$$EI_{\zeta} \frac{d^2V}{dx^2} - \beta_1 \frac{dV}{dx} = 0 \quad \text{at } x = 0$$

$$EI_{\zeta} \frac{d^2V}{dx^2} + \beta_1 \frac{dV}{dx} = 0 \quad \text{at } x = \ell$$

$$EI_{\eta} \frac{d^2W}{dx^2} - \beta_1 \frac{dW}{dx} = 0 \quad \text{at } x = 0$$

$$EI_{\eta} \frac{d^2W}{dx^2} + \beta_1 \frac{dW}{dx} = 0 \quad \text{at } x = \ell$$

$$\frac{d^2\Phi}{dx^2} - k_2 \frac{d\Phi}{dx} = 0 \quad \text{at } x = 0 \text{ and } x = \ell$$

$$Gc \frac{d\Phi}{dx} + Ec_w \frac{d^3\Phi}{dx^3} - \beta_2 \Phi = 0 \quad \text{at } x = 0$$

$$Gc \frac{d\Phi}{dx} + Ec_w \frac{d^3\Phi}{dx^3} + \beta_2 \Phi = 0 \quad \text{at } x = \ell$$

Twelve homogeneous equations are obtained when the solution for V , W , and Φ are substituted into the above boundary conditions. For a nontrivial solution the determinant of the coefficients A_i must be zero. This determinant will be called the boundary condition determinant. When the solutions to V , W and Φ are in the form of equations 2.29 the boundary condition determinant will be of the following form

$EI_\zeta r_1^3 + k_1$	$-[EI_\zeta r_1^3 - k_1]$	k_1	$-EI_\zeta r_2^3$
$r_1^\ell [EI_\zeta r_1^3 + k_1]$	$-r_1^\ell [EI_\zeta r_1^3 - k_1]$	$-e [EI_\zeta r_1^3 - k_1]$	$-EI_\zeta r_2^3 \sin r_2^\ell + k_1 \cos r_2^\ell$
$\gamma_1 [EI_\eta r_1^3 - k_1]$	$-\gamma_1 [EI_\eta r_1^3 + k_1]$	$r_1^\ell [EI_\eta r_1^3 - k_1]$	$EI_\zeta r_2^3 \sin r_2^\ell + k_1 \cos r_2^\ell$
$\gamma_1 e [EI_\eta r_1^3 + k_1]$	$-\gamma_1 e [EI_\eta r_1^3 + k_1]$	$\gamma_1 e [EI_\eta r_1^3 - k_1]$	$-EI_\zeta r_2^3 \sin r_2^\ell - k_1 \cos r_2^\ell$
$EI_\zeta r_1^2 - \beta_1 r_1$	$EI_\zeta r_1^2 + \beta_1 r_1$	$r_1^\ell [EI_\zeta r_1^2 + \beta_1 r_1]$	$-EI_\zeta r_2^3$
$r_1^\ell [EI_\zeta r_1^2 + \beta_1 r_1]$	$-r_1^\ell [EI_\zeta r_1^2 - \beta_1 r_1]$	$e [EI_\zeta r_1^2 - \beta_1 r_1]$	$-r_1^\ell [EI_\zeta r_1^2 \cos r_2^\ell - \beta_1 r_2 \sin r_2^\ell]$
$\gamma_1 (EI_\eta r_1^2 - \beta_1 r_1)$	$\gamma_1 [EI_\eta r_1^2 + \beta_1 r_1]$	$\gamma_1 e [EI_\eta r_1^2 + \beta_1 r_1]$	$-EI_\zeta r_2^2 \sin r_2^\ell + \beta_1 r_2 \cos r_2^\ell$
$r_1^\ell [EI_\eta r_1^2 + \beta_1 r_1]$	$-r_1^\ell [EI_\eta r_1^2 - \beta_1 r_1]$	$\gamma_1 e [EI_\eta r_1^2 - \beta_1 r_1]$	$-EI_\zeta r_2^2 \cos r_2^\ell - \beta_1 r_2 \sin r_2^\ell$
$\psi_1 [r_1^2 - k_2 r_1]$	$\psi_1 [r_1^2 + k_2 r_1]$	$r_1^\ell [r_1^2 - k_2 r_1]$	$\psi_1 e [r_1^2 + k_2 r_1]$
$\psi_1 [Gcr_1 + Ec_w r_1^3 - \beta_2]$	$\psi_1 [-Gcr_1 - Ec_w r_1^3 - \beta_2]$	$r_1^\ell [Gcr_1 + Ec_w r_1^3 + \beta_2]$	$\psi_1 e [-Gcr_1 - Ec_w r_1^3 + \beta_2]$
$\psi_1 e [Gcr_1 + Ec_w r_1^3 + \beta_2]$	$\psi_1 e [-Gcr_1 - Ec_w r_1^3 + \beta_2]$	$\psi_2 [Gcr_2 \cos r_2^\ell - Ec_w r_2^3 \cos r_2^\ell + \beta_2 \cos r_2^\ell]$	$\psi_2 [Gcr_2 \cos r_2^\ell - Ec_w r_2^3 \cos r_2^\ell - \beta_2 \sin r_2^\ell]$

If a natural frequency p_n is known the characteristic equation can be solved for the roots r_i . When these roots are substituted into the boundary condition determinant its value will be equal to zero. The use of the boundary condition determinant and the characteristic equation suggests a trial and error method of solving for the natural frequencies of a member. First a value of thrust is selected and a value of frequency is assumed. These are substituted into the characteristic equation, which is then solved for its roots. These roots are substituted into the boundary condition determinant, and the determinant is evaluated. If the value is not equal to zero then another frequency is assumed until one is found which makes the determinant equal to zero. This frequency is a natural frequency of the member. A digital computer was employed to facilitate the calculations.

This method of solution was used to solve the specific problem of an angle member with an unequal leg cross section with the bending moments at the ends resisted elastically and the rest of the supports the same as fixed ends. The boundary conditions were taken as symmetrical.

2.5 SOLUTION FOR MEMBER OF UNEQUAL LEG ANGLE

For an angle member with an unequal leg cross section the value of the warping constant c_w is zero, since the cross section is of thin rectangular elements with a common point of intersection as stated by Timoshenko and Gere (12). When the warping constant is zero the governing differential equation is reduced to tenth order, so the characteristic equation is reduced to tenth degree. The characteristic equation is

$$\begin{aligned}
& \frac{E^2 I_{\zeta} I_n}{m^3 A^2} \left\{ \frac{P}{A} - \frac{Gc}{I_0} \right\} r^{10} + \frac{E}{m^2 A^2} \left[-\frac{p_n^2 E I_{\zeta} I_n}{A} + \frac{P^2 I_p}{A} - \frac{P^2}{I_0} (c_z^2 I_n + c_y^2 I_{\zeta}) \right. \\
& \left. - \frac{PGcI_p}{I_0} \right] r^8 + \left(\frac{P^2}{m} E \left[\frac{PI_p}{AI_0} - \frac{Gc}{I_0} \right] - \frac{p_n^2 E}{m^2 A} \left\{ 2P \left[\frac{I_p}{A} - \frac{1}{I_0} (c_z^2 I_n + c_y^2 I_{\zeta}) \right] \right. \right. \\
& \left. \left. - \frac{GcI_p}{I_0} \right\} \right) r^6 + \left\{ \frac{p_n^4 E}{m} \left[\frac{I_p}{A} - \frac{1}{I_0} (c_z^2 I_n + c_y^2 I_{\zeta}) \right] - \frac{p_n^2 P}{m^2 A} \left[\frac{3PI_p}{I_0 A} - \frac{2Gc}{I_0} \right] \right\} r^4 \\
& \left. + \frac{p_n^4}{m} \left[\frac{3PI_p}{I_0 A} - Gc \right] r^2 - \frac{p_n^6 I_p}{I_0} = 0 \right. \quad 2.30
\end{aligned}$$

The values of ψ and γ are not affected by the value of c_w .

The size of the boundary condition determinant is reduced by the fact that c_w is zero. Since the characteristic equation is of tenth degree there are only ten roots; therefore, two columns are dropped from the determinant. When c_w is zero it means that $(\bar{w}_s - w_s)$ is equal to zero; therefore, the equations describing the warping displacements of the cross section are identically equal to zero and cannot be used as boundary conditions. This reduces the number of rows in the boundary condition determinant by two. When c_w is equal to zero as it is with an unequal leg angle cross section the boundary condition determinant is reduced to ten rows and ten columns.

The boundary conditions used were those thought to most closely approximate the boundary conditions of the experimental work done in this thesis. Since there was no translation at the supports the value of k_1 was taken to be infinity. The supports were assumed to allow no rotation about the longitudinal axis so the value of β_2 was set equal to infinity.

Since comparison of experimental results with theoretical results for fixed supports showed that the supports were elastic, the value of β_1 was taken non zero and finite. The ten resulting boundary conditions used were

$$V = W = \Phi = 0 \quad \text{at } x = 0 \text{ and } x = \ell$$

$$EI_{\zeta} \frac{d^2V}{dx^2} - \beta \frac{dV}{dx} = 0 \quad \text{at } x = 0$$

$$EI_{\zeta} \frac{d^2V}{dx^2} + \beta \frac{dV}{dx} = 0 \quad \text{at } x = \ell \quad 2.31$$

$$EI_{\eta} \frac{d^2W}{dx^2} - \beta \frac{dW}{dx} = 0 \quad \text{at } x = 0$$

$$EI_{\eta} \frac{d^2W}{dx^2} + \beta \frac{dW}{dx} = 0 \quad \text{at } x = \ell$$

The determinant which results from using these boundary conditions is

1	0	0	$\sin r_2^\lambda$	2.32
$e^{r_1^\lambda}$	$-r_1^\lambda$	$\cos r_2^\lambda$		
$\gamma_1 e^{r_1^\lambda}$	$\gamma_1 e^{-r_1^\lambda}$	$\gamma_2 \cos r_2^\lambda$		
$\psi e^{r_1^\lambda}$	$\psi e^{-r_1^\lambda}$	$\psi_2 \cos r_2^\lambda$		
ψ	ψ	$\psi_2 \sin r_2^\lambda$		
$EI_\zeta r_1^2 - \beta r_1$	$EI_\zeta r_1^2 + \beta r_1$	$-EI_\zeta r_2^2$	$-\beta r_2$	
$r_1^\lambda (EI_\zeta r_1^2 + \beta r_1)$	$e^{-r_1^\lambda} (EI_\zeta r_1^2 - \beta r_1)$	$-EI_\zeta r_2^2 \cos r_2^\lambda + \beta r_2 \sin r_2^\lambda$	$-EI_\zeta r_2^2 \sin r_2^\lambda + \beta r_2 \cos r_2^\lambda$	
$\gamma_1 (EI_\eta r_1^2 - \beta r_1)$	$\gamma_1 (EI_\eta r_1^2 + \beta r_1)$	$-\gamma_2 EI_\eta r_2^2$	$-\gamma_2 \beta r_2$	
$r_1^\lambda (EI_\eta r_1^2 + \beta r_1)$	$r_1^\lambda (EI_\eta r_1^2 - \beta r_1)$	$-\gamma_2 (EI_\eta r_2^2 \cos r_2^\lambda - \beta r_2 \sin r_2^\lambda)$	$-\gamma_2 (EI_\eta r_2^2 \sin r_2^\lambda - \beta r_2 \cos r_2^\lambda)$	

The trial and error solution outlined in the previous section was used to solve 2.30 and 2.32 for the three lowest natural frequencies. An IBM 360 computer was programmed in the Fortran IV language to carry out the necessary computations. The data read into the program were the values of thrust and frequency squared. The program was run once for each value of β . The results were plotted as thrust versus the square of the natural frequency, with the value of β used as a parameter.

CHAPTER III

EXPERIMENTAL APPARATUS AND PROCEDURE

The experiments conducted were designed to test the effect of different support conditions upon the natural frequencies of a single span thin walled member of open cross section with an applied axial load. The value of the axial load was varied from near the compressive buckling load of the member to ten thousand pounds tension, with the three lowest natural frequencies being determined for each load. The test member was an unequal leg angle which was bolted by various methods to a fixed support at one end and to a sliding support at the other. A hydraulic pump and ram system was used to move the sliding support and thus applied the axial load to the member. The strain caused by the axial load was measured by electrical resistance strain gauges and used to calculate the axial load. Then free vibrations were incited in the member and were measured by recording strain gauge outputs through the use of an oscilloscope.

3.1 TEST MEMBER AND SUPPORTS

The member used in the experiments was a mild steel $1\frac{1}{2} \times 2 \times \frac{1}{8}$ angle ninety inches long. The nominal thickness agreed with the actual measured value. When the member was clamped in the apparatus it was straight.

The different methods of supporting the member for the tests are shown in figures 3.1, 3.2, and 3.3. Figure 3.1 shows the experimental approximation of fixed supports. The member is clamped between six inch

lengths of mild steel having angular cross sections and equal legs one half inch thick. For the upper piece of the support the angle has three inch legs while the lower piece has three and one half inch legs. The clamps were milled to fit the member. The member was placed in the clamps as shown. Figure 3.2 shows the member bolted to the support with the long leg being bolted down. The short leg of the angle is shown bolted down in figure 3.3. The last two methods of supporting the member are the most common for members of angular cross section, but do not approximate fixed ends as well as the method shown in figure 3.1.

3.2 APPARATUS

The apparatus used in the experiments is diagrammed in detail in figure 3.4.

An I-beam twenty feet long, two feet deep with nine inch wide flanges was used as a base which, compared with the test member, could be considered rigid. The apparatus for supporting the test member was fixed to this base as shown in figure 3.4.

Figure 3.4 shows the apparatus used for applying the axial load. The hydraulic ram is mounted in the slide and yoke system. The base of the support is welded onto a three quarter inch plate which has its edges milled to fit grooves milled out of one and one half inch bars. The bars are welded to the I-beam. The grooves were greased to decrease friction. The apparatus was designed so that the ram could be mounted to put the test member into either tension or compression. The volume of fluid in the ram was kept constant at any desired level by shutting off a valve between

the pump and ram. This was done to maintain a constant value of the pressure and thus the axial load while the vibration records were being taken.

The strain caused by the applied axial load was measured by eight Baldwin-Lima-Hamilton SR-4 strain gauges. The gauges were positioned three inches from the support attached to the slide when the beam was clamped and when the small leg was bolted to the base of the clamp. The gauges were three inches from the fixed end when the long leg was bolted to the base of the clamp. The value of the axial load was calculated from the strain in the gauges. In the calculation of the load the value of Young's modulus was assumed to be 29.5×10^6 psi. The same test specimen was used by Didrikson (9) who showed that the gauges used were adequate to accurately measure the axial strain.

Two Baldwin-Lima-Hamilton SR-4 strain gauges were used to measure the strain caused by the vibrations of the member. The gauges were placed forty-five inches from the end of the test member and at the extremity of each leg. The electrical signals from the gauges were amplified by an Ellis Associates BAM-1 Bridge Amplifier and Meter. The signal from this was recorded using a Brush Oscillograph, model 16-2308-00.

The axial strain was measured by the BAM-1 which has 1% full scale accuracy. The frequency response of the BAM-1 was 0 - 20,000 cycles per second within 5%. The frequency response for the Brush Oscillograph was $\pm 5\%$ for frequencies to 1000 cps.

3.3 EXPERIMENTAL PROCEDURE

For each of the tests the member was loaded in thousand pound increments from zero to ten thousand pounds in tension and from zero to near the buckling load in compression.

First a zero reading was taken for the strain gauges used for measuring the axial strain. Then the load was applied until the gauges registered the amount of strain caused by the desired axial load. Next the beam was struck in such a way that one of the two lowest modes of vibration was excited and recorded. The axial strain was then read again and the load was released. Then the zero reading was checked for drift. This procedure was repeated for each load.

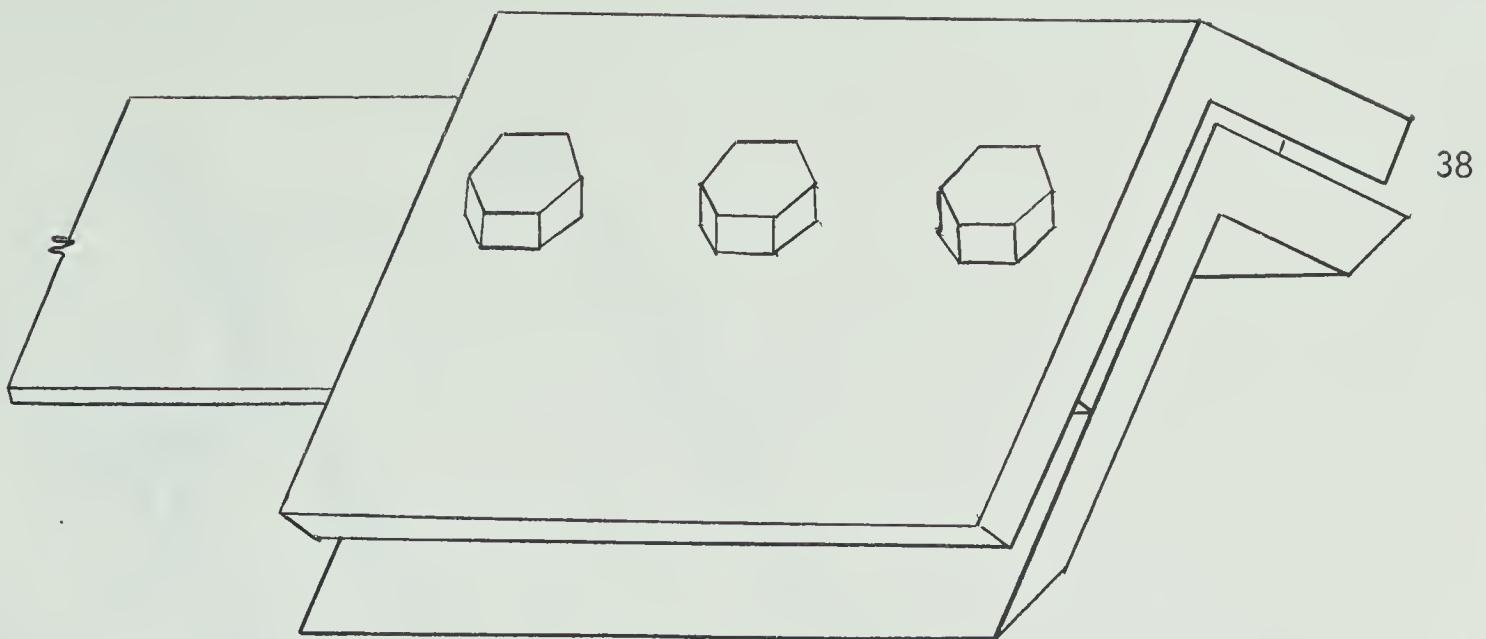


Figure 3.1 BOTH LEGS CLAMPED - OBLIQUE VIEW

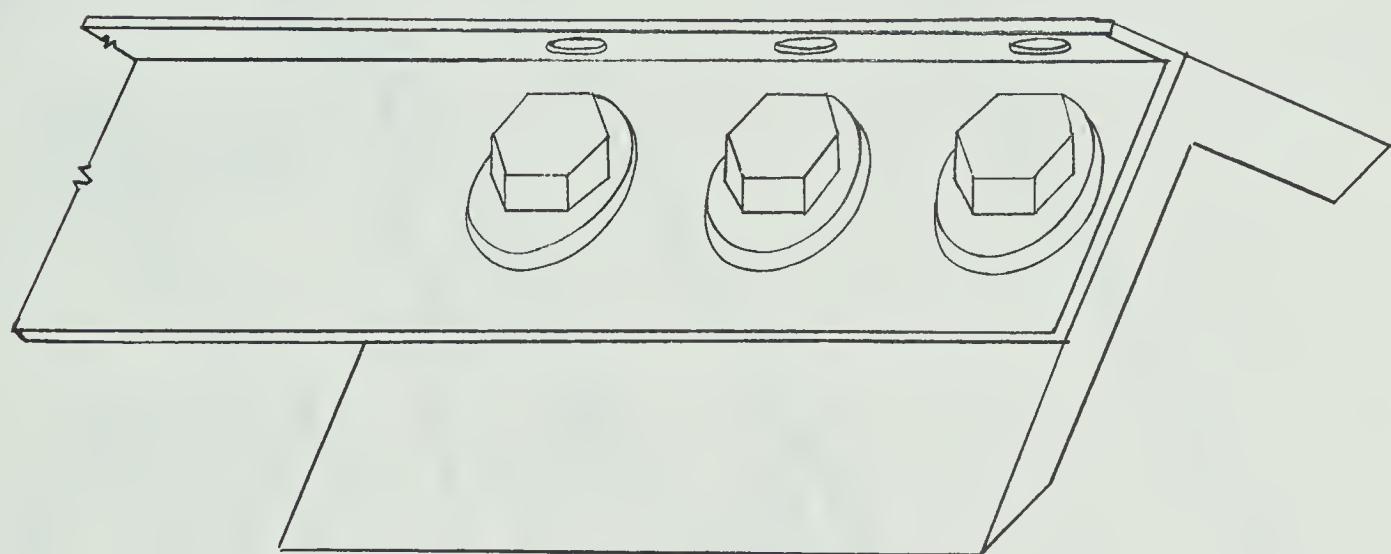


Figure 3.2 LONG LEG CLAMPED - OBLIQUE VIEW

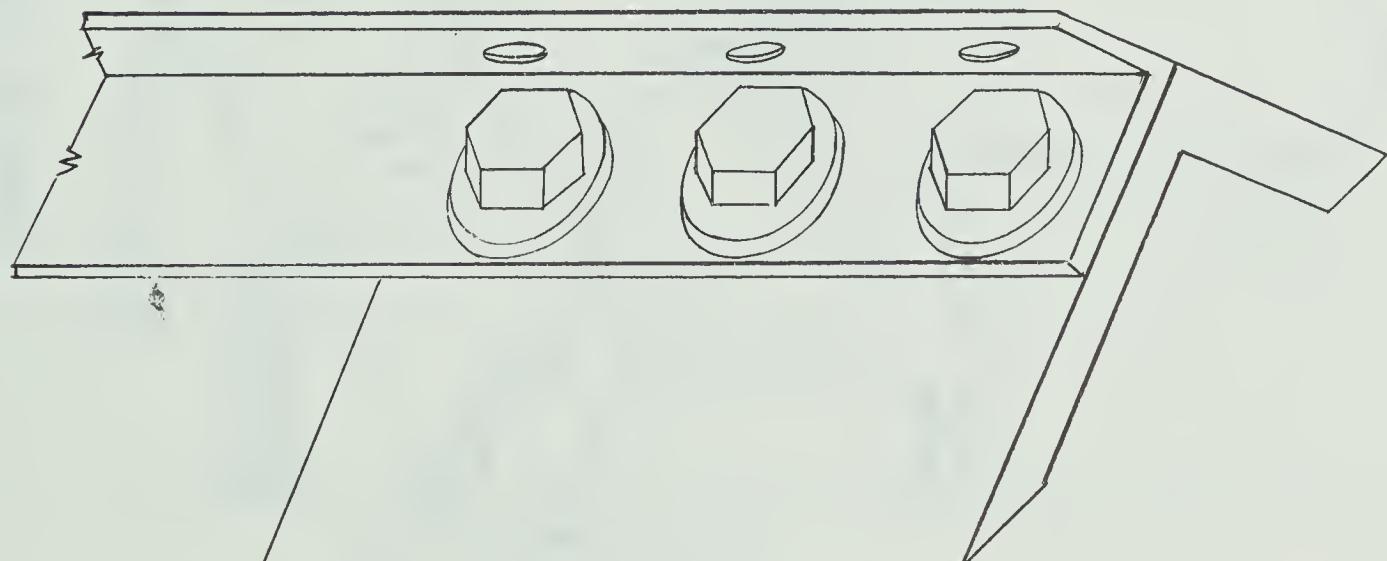


Figure 3.3 SHORT LEG CLAMPED - OBLIQUE VIEW

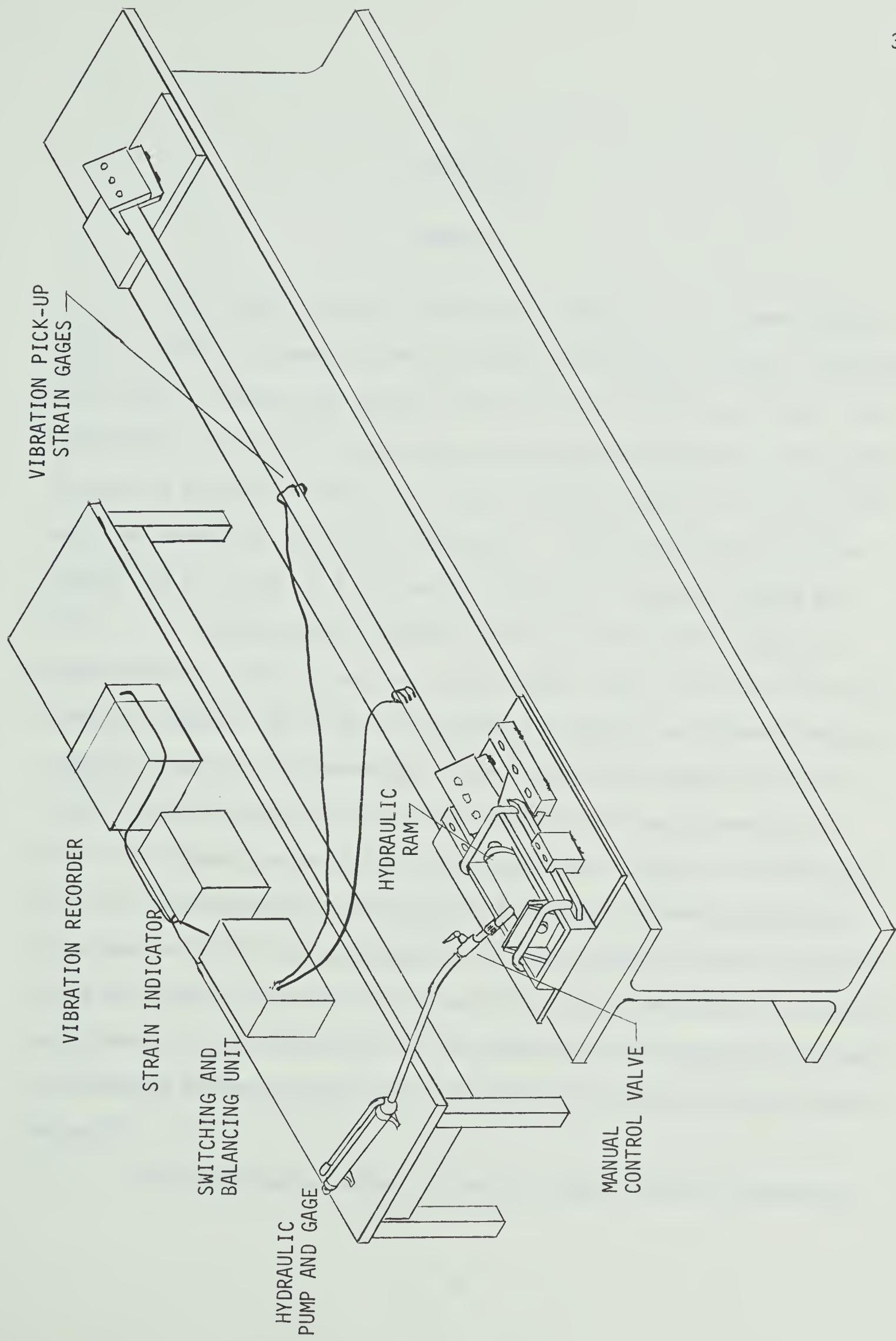


Figure 3.4 SCHEMATIC VIEW OF EXPERIMENTAL APPARATUS (NOT TO SCALE)

CHAPTER IV

RESULTS

This chapter presents the results obtained in the investigation into the effect of boundary conditions upon the coupled torsional vibrations of an axially loaded angle member with an unequal leg cross section. The experimental results are compared with the theoretical results, which were obtained by programming the University's IBM-360 computer to do the trial and error procedure outlined in Chapter II. The required input for the program was one value of β and various values of frequency squared and thrust. The program results gave the value of the boundary condition determinant for a particular set of input data. When the sign of the determinant changed, the value of the natural frequency was between the two values of frequency corresponding to the determinant values of different sign. The difference could be made as small as desired by reading the values of frequency squared into the program with as small a difference as desired. The difference of the values of frequency squared used for the fixed ends was $100 \text{ (radians/second)}^2$. For the accuracy to which the graphs could be plotted a solution this close to the actual value was not required, so in order to save computing time the remaining results were obtained with a difference between the two values of frequency squared of $1000 \text{ (radians/second)}^2$.

The experimental values of the two lowest natural frequencies

for the values of thrust used were easily obtained by the experimental procedures outlined in the third chapter. The frequencies higher than the second were difficult to obtain because of the predominance of the lower two frequencies of the member. No suitable method could be found to excite only the higher frequencies. For this reason the theoretical results presented in this chapter are the first three frequencies for the member with fixed ends and only the first two frequencies for the beam with elastic end conditions.

The frequencies measured experimentally resulted from different modes of vibration of the member. This was evident because they were obtained by striking the member from various angles so that different amounts of the three displacements were present in the vibration.

4.1 FIXED ENDS

The theoretical results obtained by using fixed end boundary conditions are presented in table 4.1 and figures 4.1 and 4.2. The boundary condition determinant used for fixed ends is presented in Appendix 1. These results are presented to complete the theoretical results of Didrikson (9). The program used in the previous work did not give results for all values of thrust for the second and third modes. The determinant solution was changed so that the frequencies of vibration could be obtained for any value of thrust for the first three modes. The curves shown are those of the first two fundamental modes of vibration on figure 4.1 and the third mode on figure 4.2. The theoretical buckling load of the member is the value of thrust for which the frequency of the

lowest natural mode is zero.

The results of the test which most closely approximates clamped ends are given in table 4.2. These results are compared with the theoretical results for the first two fundamental modes in figure 4.3. This figure shows that the experimental values lie on a straight line which is lower than the plot of the theoretical values and has a smaller slope. The experimental line when extrapolated crosses the thrust axis at a value lower than the theoretical buckling load. Didrikson (9) showed that the variation of the theoretical curves caused by varying the values of the modulus of elasticity E and the modulus of rigidity G would not account for the differences between the theoretical and experimental values of the frequencies squared. When the moduli are varied the plotted line of frequency squared versus thrust is moved up or down but the slope of the line does not change. The differences between the theoretical and experimental curves are explained by taking the end conditions to be elastic instead of fixed ends.

4.2 ELASTIC END CONDITIONS

The theoretical results discussed in this section are those for which the boundary conditions are expressed in equation 2.31. For those end conditions the values of the spring constant β were taken as a parameter and various curves of frequency squared versus thrust were obtained for the first two modes by using the procedure outlined in chapter II. The value of β was varied from zero, to infinity, which corresponds to fixed supports. Table 4.3 gives values for and figure 4.4 shows representa-

tive curves of thrust versus frequency squared obtained for the various values of β for the first mode of vibration. Similarly, table 4.4 and figure 4.5 give the theoretical results for the second natural frequency. From figure 4.4 and 4.5 it can be seen that the larger the value of the equivalent spring constant β , the greater the slope of the curve, and the higher the point where the curve crosses the frequency squared axis.

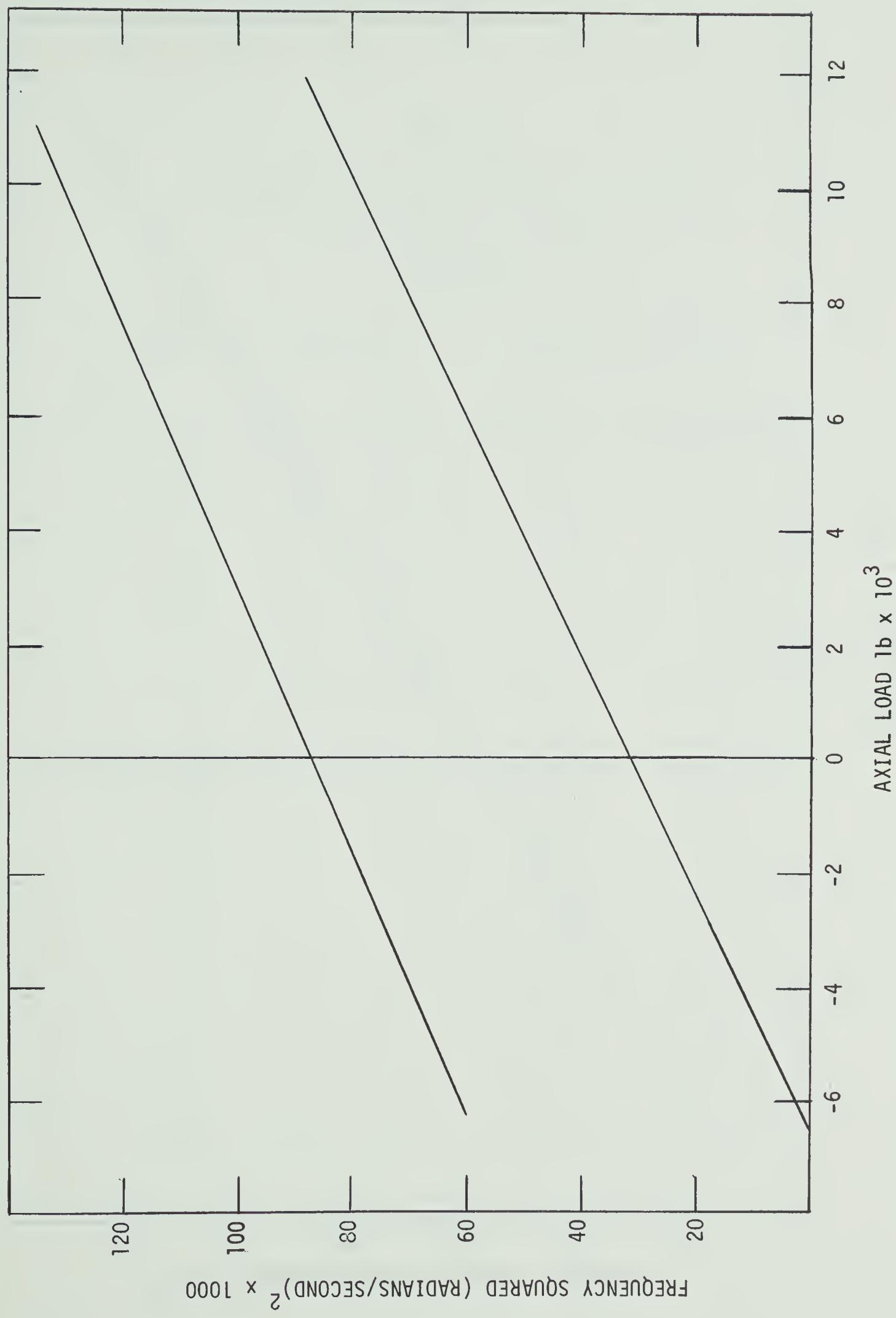
The points shown in figures 4.6, 4.7, and 4.8 show the experimental results for the various end conditions. These were used to establish effective values of β for the first two modes. The theoretical curves for the values of β that best fit the experimental results are shown in the same figures, and can be seen to fit the points very well. The small deviations are probably due to errors in the experimental method.

Since the value of the effective spring constant β is smaller for the first mode of vibration than for the second mode it appears that the value of β depends upon the mode of vibration.

TABLE 4.1
CALCULATED NATURAL FREQUENCIES OF ANGLE UNDER AXIAL LOAD
FIXED ENDS

THRUST lb.	FREQUENCY			FREQUENCY SQUARED		
	1 st cps	2 nd cps	3 rd cps	1 st $(r/sec)^2$	2 nd $(r/sec)^2$	3 rd $(r/sec)^2$
-6,000	7.1	39.5	82.5	2,000	61,400	268,000
-4,000	17.4	42.1	84.5	12,000	70,000	283,800
-2,000	23.6	44.6	86.8	22,000	78,000	297,700
0	28.1	47.0	88.4	31,200	87,200	308,800
2,000	32.3	50.8	90.3	41,200	95,900	322,500
4,000	35.8	51.4	92.0	50,700	104,200	333,900
6,000	39.0	53.5	93.6	60,000	113,000	345,000
8,000	45.0	54.8	94.8	69,200	118,200	355,000
10,000	49.4	57.6	98.0	78,200	131,000	366,000

Figure 4.1 CALCULATED FREQUENCY SQUARED vs AXIAL LOAD - FIRST AND SECOND MODE - FIXED ENDS



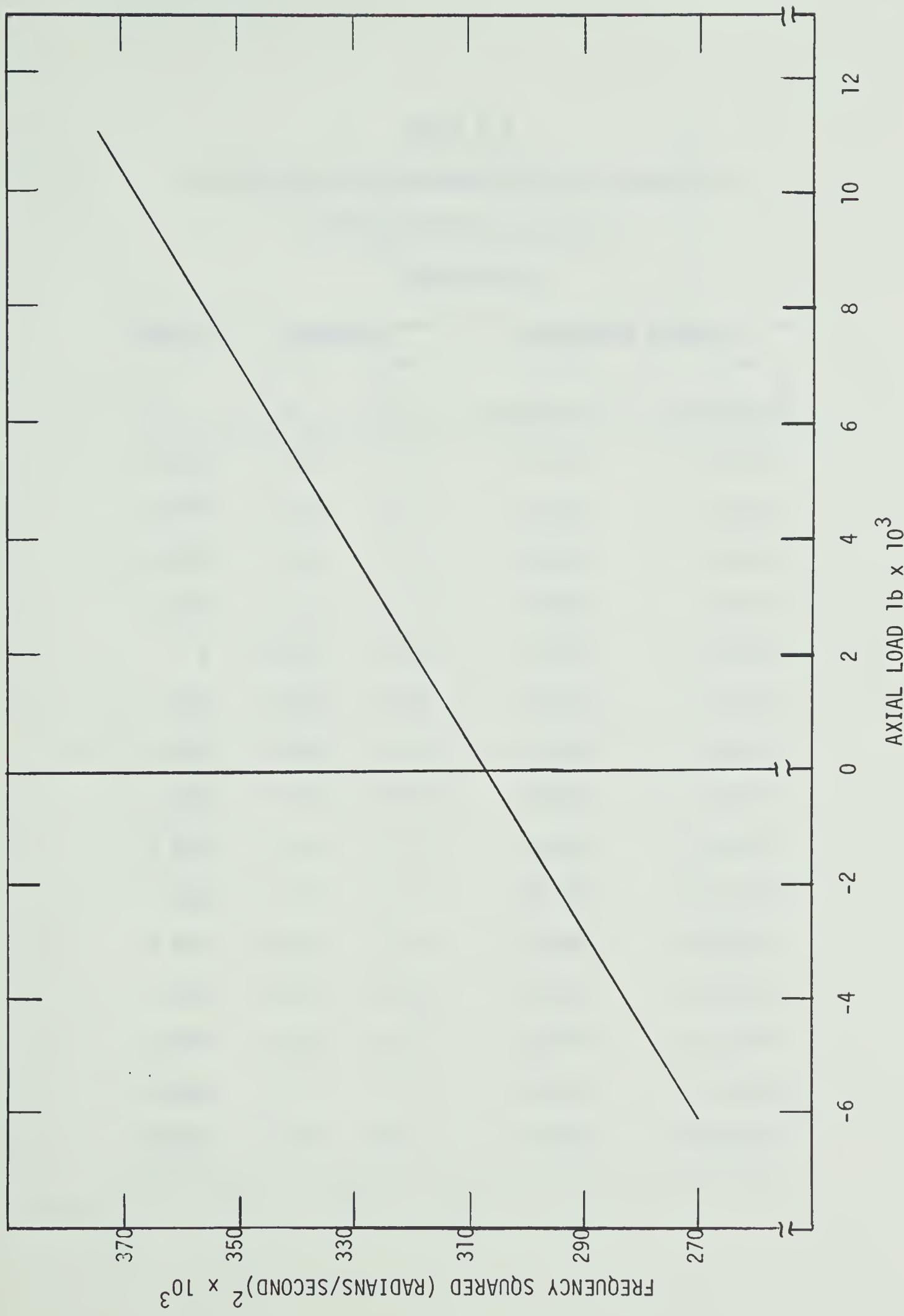


Figure 4.2 CALCULATED FREQUENCY SQUARED vs AXIAL LOAD

THIRD MODE — FIXED ENDS

TABLE 4.2
 EXPERIMENTALLY DETERMINED NATURAL FREQUENCIES
 OF ANGLE UNDER AXIAL LOAD
 CLAMPED ENDS

THRUST lb.	FREQUENCY		FREQUENCY SQUARED	
	1 st cps	2 nd cps	1 st (rad/sec) ²	2 nd (rad/sec) ²
-4,000	15.7	41.5	9,720	68,100
-3,000	19.3	42.3	14,700	70,700
-2,000	21.8	43.5	18,800	75,000
-1,000	24.0	44.7	22,800	79,000
0	26.6	45.6	27,900	82,000
1,000	29.0	46.9	33,500	86,500
2,000	30.8	47.9	37,500	90,600
3,000	31.9	48.8	40,200	93,600
4,000	34.0	50.0	45,800	98,600
5,000	35.4	51.0	49,500	103,000
6,000	36.8	51.8	53,500	106,400
7,000	38.7	52.5	59,300	109,000
8,000	39.9	53.7	63,000	114,200
9,000	41.1	55.6	66,600	122,500
10,000	42.6	56.1	71,900	125,000

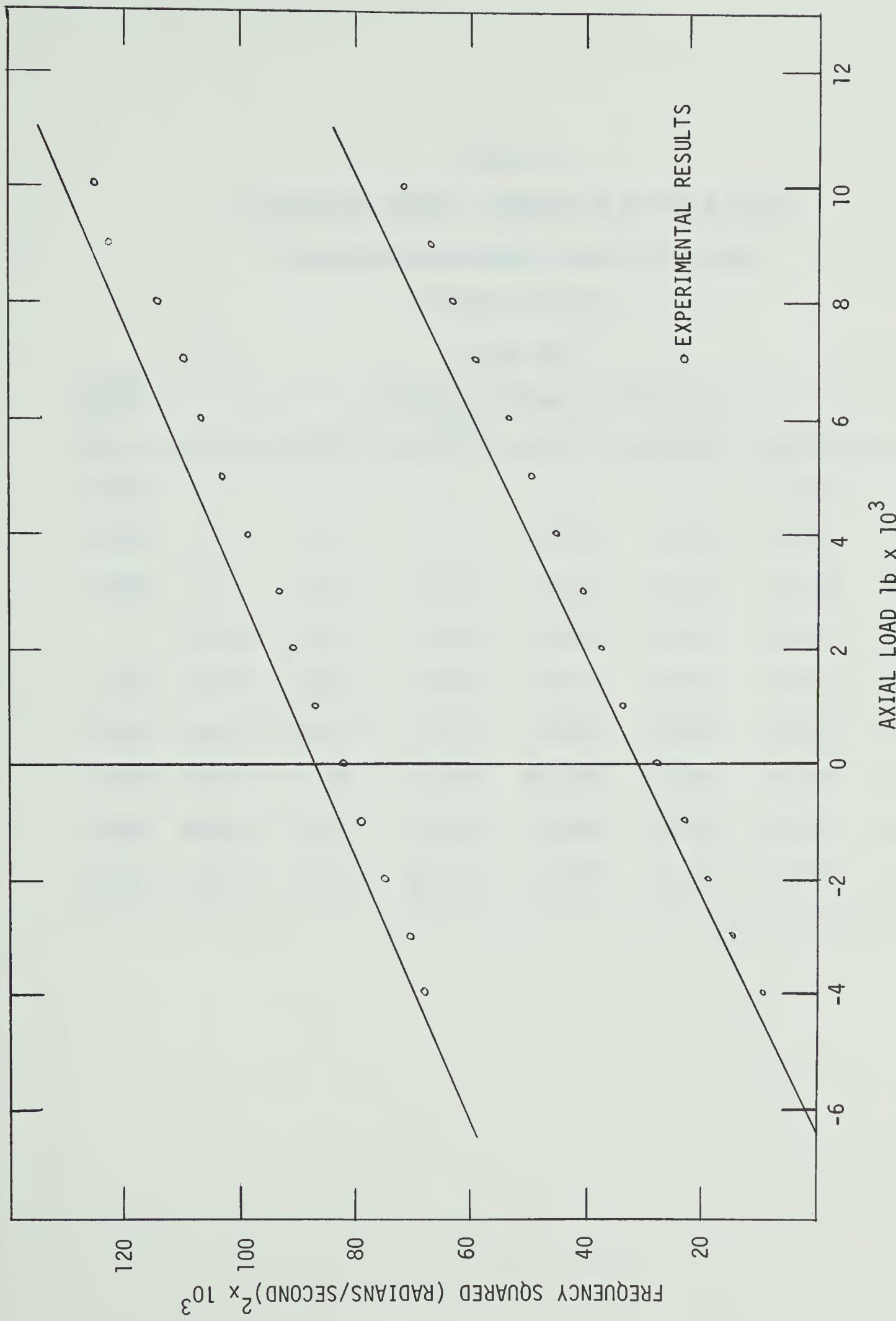


Figure 4.3 PLOT OF CALCULATED FREQUENCY SQUARED vs AXIAL LOAD COMPARED WITH EXPERIMENTAL RESULTS

FIXED ENDS

TABLE 4.3
 CALCULATED NATURAL FREQUENCIES OF ANGLE UNDER
 AXIAL LOAD FOR VARIOUS VALUES OF SUPPORT
 SPRING CONSTANTS

THRUST		FREQUENCY SQUARED (RAD/SEC) ²					
lb.	$\beta = 0.0$	$\beta = 5 \times 10^4$	$\beta = 1 \times 10^5$	$\beta = 2 \times 10^5$	$\beta = 4 \times 10^5$	$\beta = 1 \times 10^6$	$\beta = \infty$
-6,000	--	--	--	--	--	1,000	2,000
-4,000	--	--	--	4,000	7,000	10,000	12,000
-2,000	--	4,000	8,000	12,000	16,000	19,000	22,000
0	7,000	12,000	17,000	21,000	25,000	29,000	31,200
2,000	15,000	20,000	25,000	29,000	34,000	38,000	41,200
4,000	23,000	29,000	33,000	38,000	43,000	46,000	50,700
6,000	30,000	37,000	41,000	46,000	51,000	56,000	60,000
8,000	38,000	45,000	50,000	54,000	60,000	63,000	69,200
10,000	46,000	53,000	58,000	63,000	68,000	73,000	78,200

TABLE 4.4
 CALCULATED NATURAL FREQUENCIES OF ANGLE UNDER
 AXIAL LOAD FOR VARIOUS VALUES OF SUPPORT
 SPRING CONSTANT
 SECOND MODE

THRUST lb.	FREQUENCIES SQUARED (RAD/SEC) ²						
	$\beta = 0.0$	$\beta = 1 \times 10^5$	$\beta = 2 \times 10^5$	$\beta = 4 \times 10^5$	$\beta = 1 \times 10^6$	$\beta = 4 \times 10^6$	$\beta = \infty$
-6,000	4,000	14,000	23,000	35,000	47,000	57,000	61,000
-4,000	12,000	22,000	31,000	43,000	55,000	65,000	70,000
-2,000	19,000	31,000	39,000	51,000	63,000	74,000	78,000
0	28,000	39,000	46,000	59,000	72,000	83,000	87,000
2,000	36,000	47,000	57,000	67,000	80,000	91,000	96,000
4,000	43,000	53,000	63,000	74,000	88,000	100,000	104,000
6,000	51,000	64,000	70,000	83,000	96,000	108,000	113,000
8,000	59,000	72,000	79,000	91,000	104,000	119,000	118,000
10,000	67,000	80,000	86,000	99,000	113,000	126,000	131,000

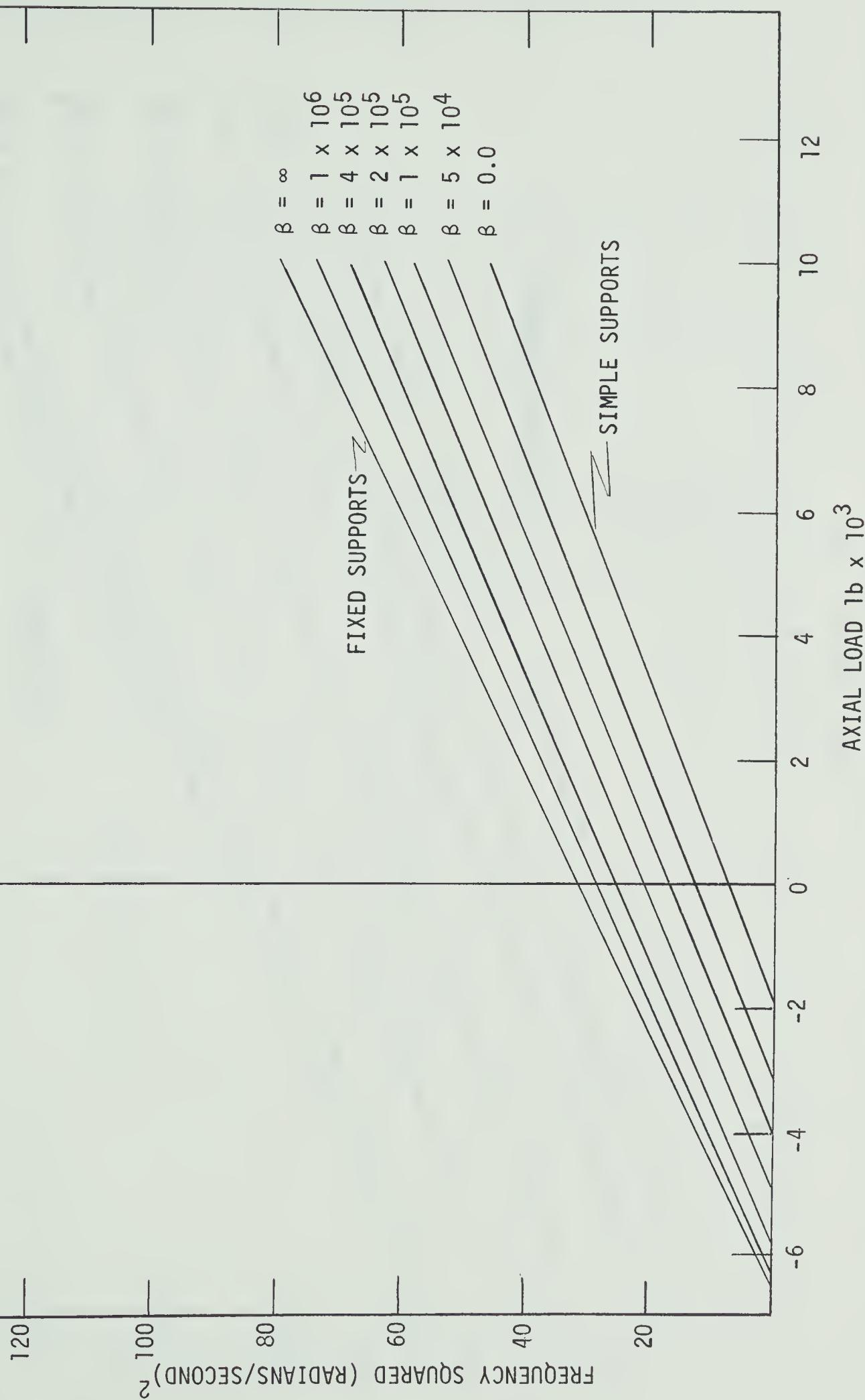


Figure 4.4 CALCULATED FREQUENCY SQUARED vs AXIAL LOAD FOR VARIOUS VALUES OF SUPPORT SPRING CONSTANTS

FIRST MODE

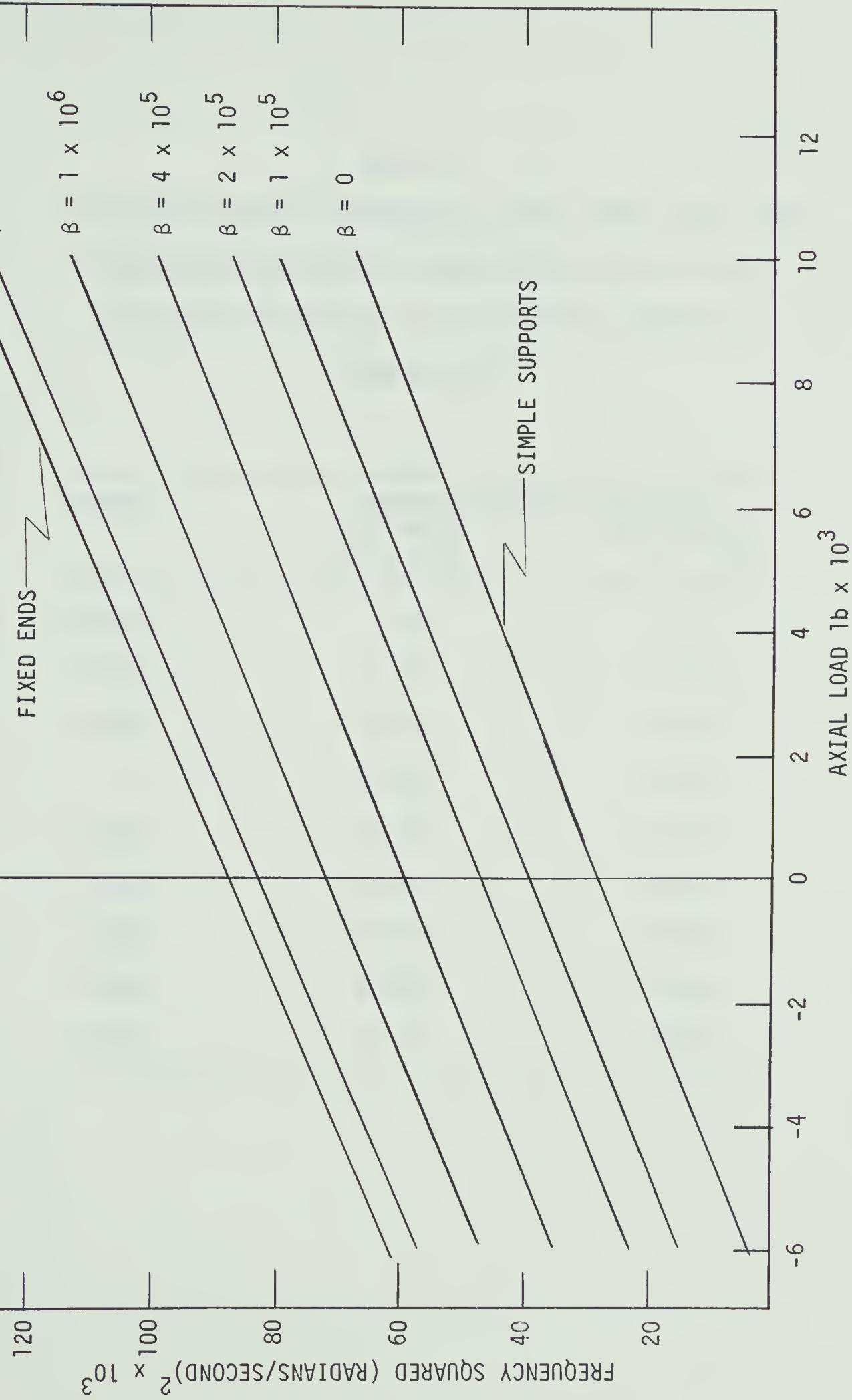


Figure 4.5 CALCULATED FREQUENCY SQUARED vs AXIAL LOAD FOR VARIOUS VALUES OF SUPPORT SPRING CONSTANTS β

SECOND MODE

TABLE 4.5
 CALCULATED NATURAL FREQUENCIES OF ANGLE UNDER AXIAL LOAD
 (SUPPORT SPRING CONSTANTS CHOSEN TO GIVE BEST FIT OF
 CALCULATED FREQUENCIES TO EXPERIMENTAL FREQUENCIES)
 CLAMPED ENDS

THRUST 1b.	FREQUENCY SQUARED FIRST MODE $\beta = 8 \times 10^5$	$(\text{RAD/SEC})^2$ SECOND MODE $\beta = 4 \times 10^6$
-6,000	500	--
-4,000	9,000	65,000
-2,000	19,000	74,000
0	27,000	82,000
2,000	37,000	91,000
4,000	46,000	100,000
6,000	55,000	108,000
8,000	64,000	117,000
10,000	73,00	125,000

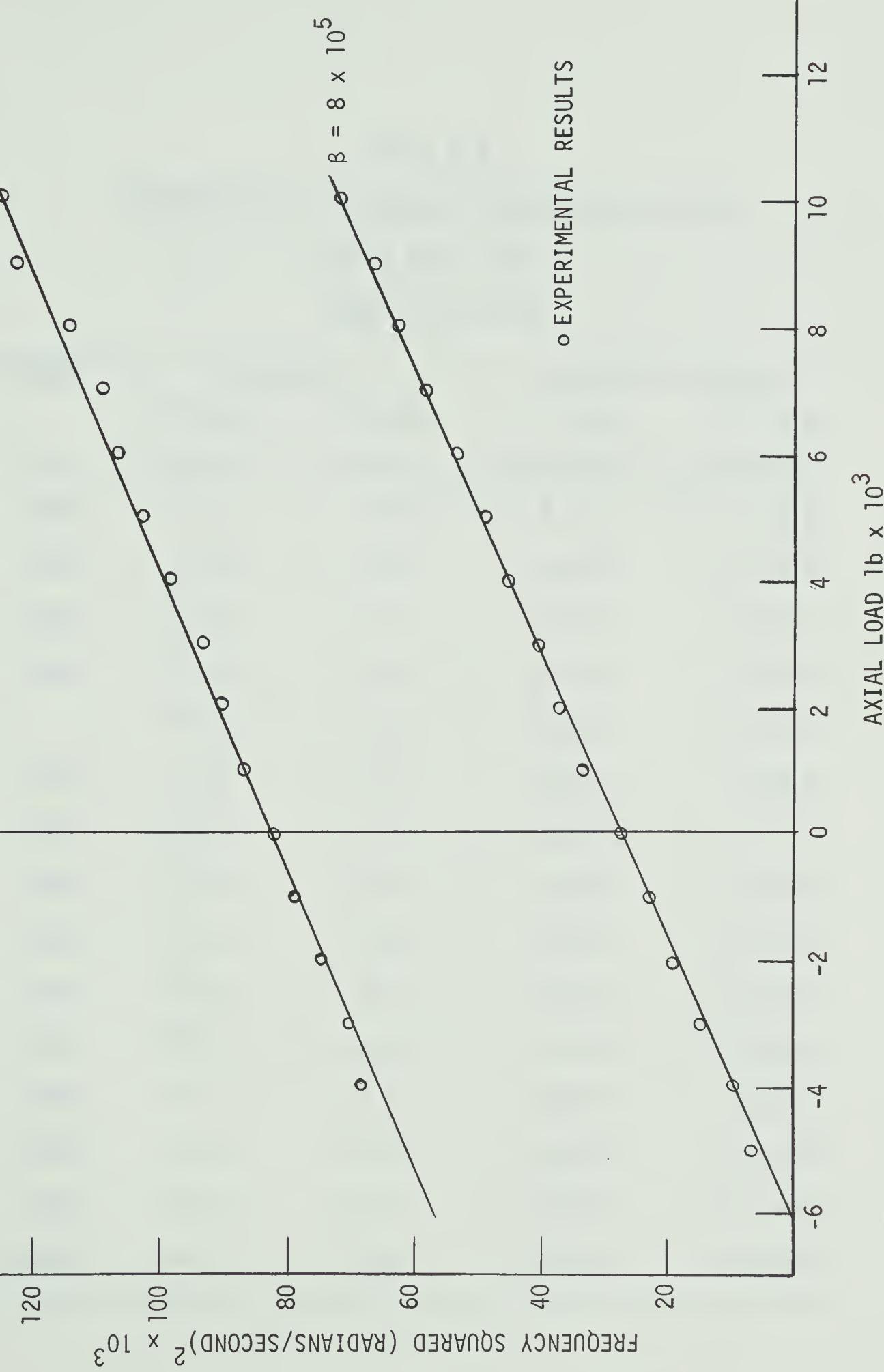


Figure 4.6 PLOT OF CALCULATED FREQUENCY SQUARED vs AXIAL LOAD WITH SUPPORT SPRING CONSTANTS CHOSEN TO GIVE BEST FIT TO EXPERIMENTAL RESULTS — CLAMPED ENDS

TABLE 4.6
 EXPERIMENTALLY DETERMINED FREQUENCIES OF ANGLE
 UNDER AXIAL LOAD
 LONG LEG CLAMPED

THRUST lb.	FREQUENCY		FREQUENCY SQUARED	
	1 st MODE cps	2 nd MODE cps	1 st MODE (rad/sec) ²	2 nd MODE (rad/sec) ²
-4,000	15.8	37.5	9,860	55,500
-3,000	18.7	39.6	13,800	61,900
-2,000	20.5	41.1	16,600	66,700
-1,000	23.2	42.3	21,300	70,600
0	24.8	42.9	24,250	72,600
1,000	26.9	44.1	28,500	76,600
2,000	30.3	45.4	36,200	81,600
3,000	31.4	46.4	38,900	84,900
4,000	33.2	47.6	43,500	89,400
5,000	35.0	49.3	48,300	95,800
6,000	36.1	50.0	51,400	98,600
7,000	37.7	51.4	56,000	104,000
8,000	38.6	52.0	58,700	106,500
9,000	40.3	53.2	64,000	111,600
10,000	41.9	56.8	69,300	127,000

TABLE 4.7
 CALCULATED NATURAL FREQUENCIES OF ANGLE UNDER AXIAL LOAD
 (SUPPORT SPRING CONSTANTS CHOSEN TO GIVE BEST FIT OF
 CALCULATED FREQUENCIES TO EXPERIMENTAL FREQUENCIES)
 LONG LEG CLAMPED

THRUST 1b.	FREQUENCY SQUARED (RAD/SEC) ²	$\beta = 1.2 \times 10^6$
	FIRST MODE $\beta = 4 \times 10^5$	
-6,000	--	49,000
-4,000	8,000	57,000
-2,000	16,000	66,000
0	25,000	74,000
2,000	34,000	82,000
4,000	43,000	91,000
6,000	52,000	99,000
8,000	60,000	107,000
10,000	68,000	116,000

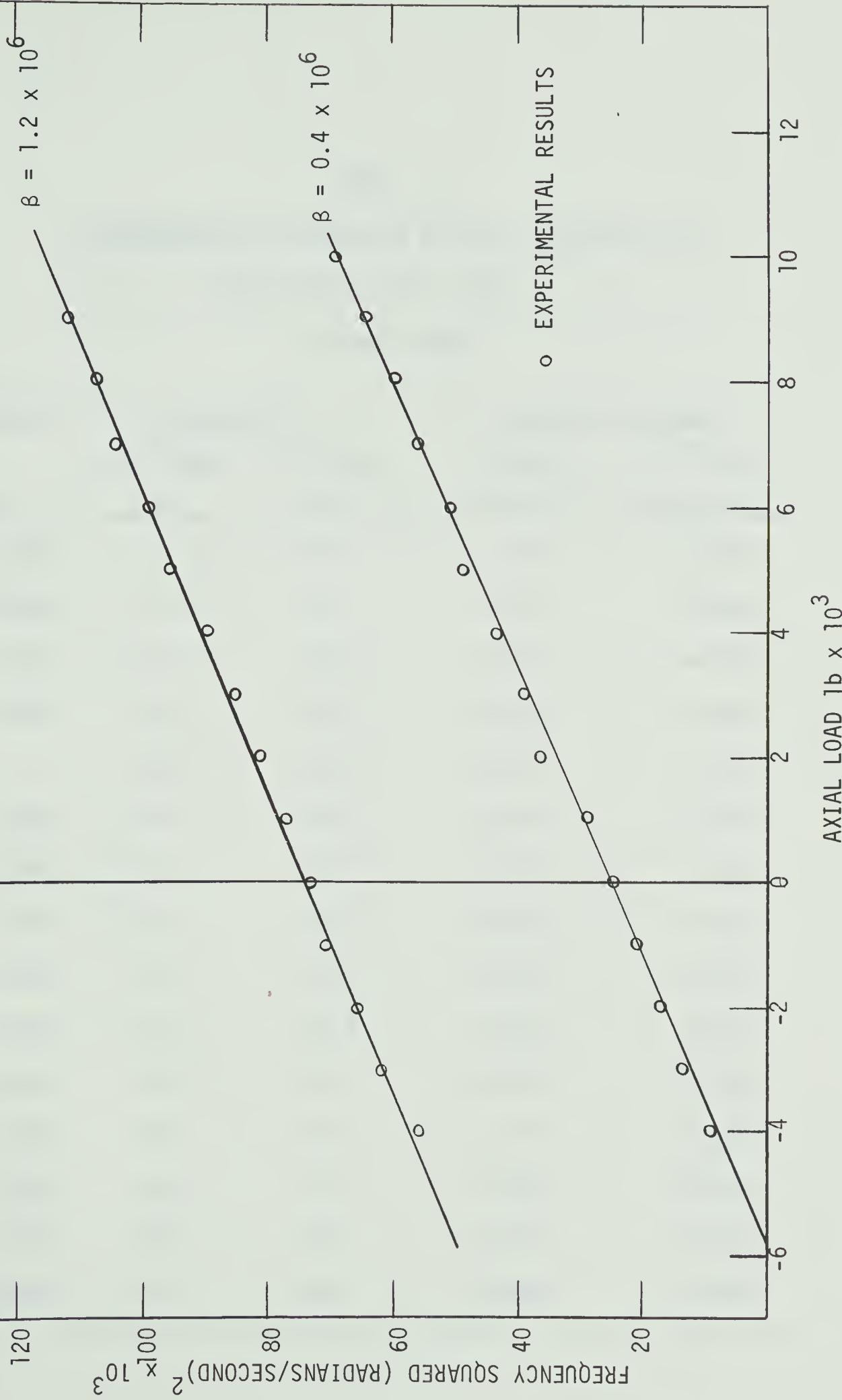


Figure 4.7 PLOT OF CALCULATED FREQUENCY SQUARED vs AXIAL LOAD WITH SUPPORT SPRING CONSTANTS CHOSEN TO GIVE BEST FIT TO EXPERIMENTAL RESULTS — LONG LEG CLAMPED

TABLE 4.8
 EXPERIMENTALLY DETERMINED NATURAL FREQUENCIES OF
 ANGLE UNDER AXIAL LOAD
 CLAMPED ENDS

THRUST 1b.	FREQUENCY		FREQUENCY SQUARED	
	1 st MODE cps	2 nd MODE cps	1 st MODE (rad/sec) ²	2 nd MODE (rad/sec) ²
-4,000	17.0	37.2	11,450	53,700
-3,000	19.6	39.1	15,100	60,600
-2,000	21.4	40.5	18,000	64,800
-1,000	23.5	41.8	21,550	69,000
0	25.7	42.8	26,100	72,400
1,000	27.7	44.1	30,300	76,700
2,000	31.0	45.3	38,000	80,900
3,000	32.0	46.4	40,600	84,600
4,000	34.0	47.3	45,800	88,200
5,000	35.0	48.3	48,400	91,900
6,000	36.5	49.6	53,800	97,400
7,000	38.1	50.7	57,100	101,000
8,000	39.3	51.9	61,000	106,200
9,000	40.4	52.7	64,500	109,600
10,000	41.7	53.6	68,600	113,500

TABLE 4.9
 CALCULATED FREQUENCIES OF ANGLE UNDER AXIAL LOAD
 SUPPORT SPRING CONSTANTS CHOSEN TO GIVE BEST FIT OF
 CALCULATED FREQUENCIES TO EXPERIMENTAL FREQUENCIES
 SHORT LEG CLAMPED

THRUST 1b.	FREQUENCY SQUARED (r/sec) ²	
	FIRST MODE $\beta = 6 \times 10^5$	SECOND MODE $\beta = 1.1 \times 10^6$
-6,000	--	--
-4,000	8,000	56,000
-2,000	18,000	64,000
0	27,000	73,000
2,000	36,000	81,000
4,000	44,000	89,000
6,000	53,000	98,000
8,000	62,000	106,000
10,000	71,000	114,000

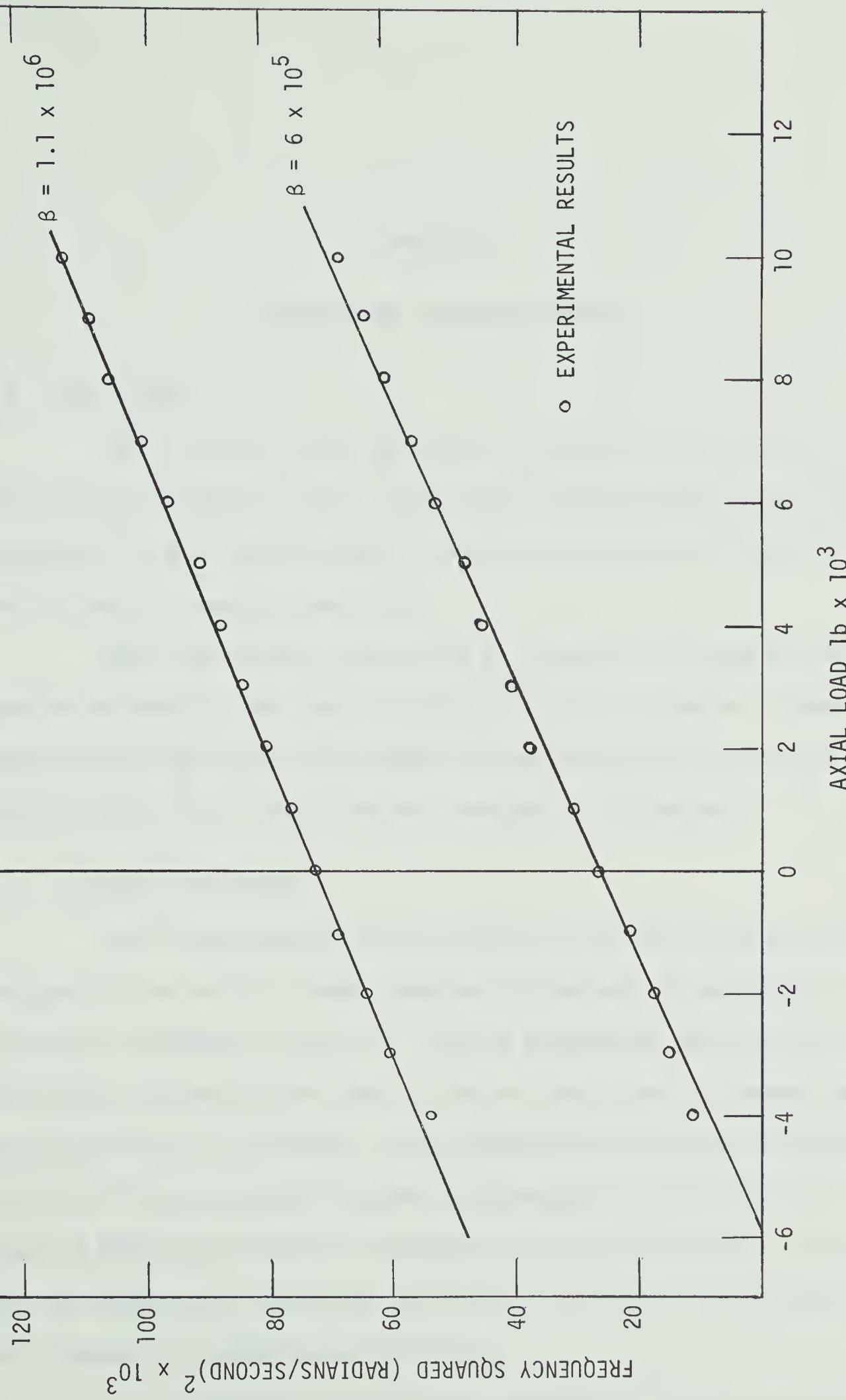


Figure 4.8 PLOT OF CALCULATED FREQUENCY SQUARED vs AXIAL LOAD WITH SUPPORT SPRING CONSTANTS CHOSEN TO GIVE BEST FIT TO EXPERIMENTAL RESULTS — SHORT LEG CLAMPED

60

CHAPTER V

CONCLUSIONS AND APPLICATIONS

5.1 CONCLUSIONS

The trial and error procedure of Gere and Lin (8) used to solve the three simultaneous partial differential equations describing the vibrations of a thin walled member of open cross section will work for any set of elastic boundary conditions.

When the boundary conditions of a member are known and are expressed in terms of the elastic spring constants of Chapter II then the axial load on the single span member can be predicted by measuring the lowest or the second lowest natural frequency of vibration.

5.2 POSSIBLE APPLICATION

This investigation of the effect of end conditions upon the frequency of an axially loaded beam was carried out in an effort to find a method of determining the axial load in a member of an existing structure. The results presented show that if the end conditions of a member are known then the thrust on the member can be determined accurately by measuring either the first or second frequency and reading the value of axial load from the theoretical curve of frequency squared versus thrust. This would give an experimental method for confirming the value of load used in designing a member in a complicated structure.

This application depends upon knowledge of the end conditions,

and in the case above one would need to know the value of β for the lowest frequency. The value of the β was seen to depend upon the mode of vibration, so the values of the spring constant would have to be found experimentally for different end conditions. An attempt was made to find a value of β statically, but the result was not the same as either of the values of β for the first two modes. Thus in order for the method outlined in this thesis to be of practical importance, it will be necessary to determine the relationship of the spring constants to the type of end support.

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APPENDIX I

In chapter II a solution was presented for the fundamental modes of vibration of an axially loaded member of general cross section. This was reduced to the specific case of an angle member with unequal leg cross section. The general case considered in Chapter II reduces to the case of a member with simple supports or clamped supports depending upon the value of the spring constant β . In this appendix the boundary condition determinant will be derived for the case of fixed ends and the problem will be solved without the use of the computer for simple supports. Both of the above solutions were used to check the results from the computer.

CLAMPED SUPPORTS

The method used to solve the problem with clamped supports is the same as the solution for general supports in chapter II, save that the proper end conditions had to be employed. This makes the boundary condition determinant slightly different. The boundary conditions for the member with both ends clamped are

$$v = w = \phi = 0 \quad \text{For } x = 0 \quad x = \ell$$

$$\frac{dv}{dx} = \frac{dw}{dx} = \frac{d\phi}{dx} = 0 \quad \text{For } x = 0 \quad x = \ell$$

This follows from the discussion of boundary conditions in the

second chapter. It has also been shown that the solutions of V , W , and Φ may be expressed in the form

$$V = A_1 e^{r_1 x} + A_2 e^{-r_1 x} + A_3^1 \cos r_2 x + A_4^1 \sin r_2 x + \dots$$

$$W = \gamma_1 A_1 e^{r_1 x} + \gamma_1 A_2 e^{-r_1 x} + \gamma_2 A_3^1 \cos r_2 x + \gamma_2 A_4^1 \sin r_2 x + \dots$$

$$\Phi = \psi_1 A_1 e^{r_1 x} + \psi_1 A_2 e^{-r_1 x} + \psi_2 A_3^1 \cos r_2 x + \psi_2 A_4^1 \sin r_2 x + \dots$$

Now to obtain the boundary condition determinant for fixed ends, these values of V , W , and Φ are substituted into the boundary conditions above and the determinant of the coefficient is set equal to zero. This determinant was earlier called the boundary condition determinant. For clamped ends it is of the form

$$\begin{vmatrix}
 1 & 1 & 1 & 0 & \dots \\
 e^{r_1 l} & e^{-r_1 l} & \cos r_2 l & \sin r_2 l & \dots \\
 \gamma_1 & \gamma_1 & \gamma_2 & 0 & \dots \\
 \gamma_1 e^{r_1 l} & \gamma_1 e^{-r_1 l} & \gamma_2 \cos r_2 l & \gamma_2 \sin r_2 l & \dots \\
 \psi_1 & \psi_1 & \psi_2 & 0 & \dots \\
 \psi_1 e^{r_1 l} & \psi_1 e^{-r_1 l} & \psi_2 \cos r_2 l & \psi_2 \sin r_2 l & \dots \\
 r_1 & -r_1 & 0 & r_2 & \dots \\
 r_1 e^{r_1 l} & -r_1 e^{-r_1 l} & 0 & r_2 & \dots \\
 \gamma_1 r_1 & -\gamma_1 r_1 & 0 & \gamma_2 r_2 & \dots \\
 \gamma_1 r_1 e^{r_1 l} & -\gamma_1 r_1 e^{-r_1 l} & -\gamma_2 r_2 \sin r_2 l & \gamma_2 r_2 \cos r_2 l & \dots \\
 \psi_1 r_1 & -\psi_1 r_1 & 0 & \psi_2 r_2 & \dots \\
 \psi_1 r_1 e^{r_1 l} & -\psi_1 r_1 e^{-r_1 l} & -\psi_2 r_2 \sin r_2 l & \psi_2 r_2 \cos r_2 l & \dots
 \end{vmatrix} = 0$$

To solve for the natural frequency of a member with clamped ends this determinant is used in the trial and error procedure outlined in chapter II. In the case where the warping constant c_w is zero the eleventh and twelfth rows and columns of the above determinant are deleted.

SIMPLE SUPPORTS

The natural frequencies of a member with simple supports could be solved for by the method outlined, but they can also be obtained by hand very easily as was done by Vlasov (2). The following shows the method used to solve the simple support case by hand. The boundary conditions are

$$V = W = \Phi = 0$$

$$\frac{d^2V}{dx^2} = \frac{d^2W}{dx^2} = \frac{d^2\Phi}{dx^2} = 0$$

These support conditions suggest the use of the solutions

$$V = V_0 \sin \frac{n\pi x}{l}$$

$$W = W_0 \sin \frac{n\pi x}{l}$$

$$\Phi = \Phi_0 \sin \frac{n\pi x}{l}$$

Substitute these values for V , W , and Φ into equations 2.22,

2.23, and 2.24 and simplify the results by observing that the natural frequencies for a simply supported member are given by

$$p_y^2 = \frac{EI}{mA} \left(\frac{n\pi}{l}\right)^4$$

$$p_z^2 = \frac{EI}{mA} \left(\frac{n\pi}{l}\right)^4$$

$$p_\phi^2 = \left(\frac{n\pi}{l}\right)^2 \left(\frac{\frac{n^2\pi^2 E c_w}{l^2 m I_0} + l^2 G c}{l^2 m I_0} \right)$$

Further simplification is made by putting

$$k_n^2 = p_n^2 + \frac{P}{mA} \left(\frac{n\pi}{l}\right)^2$$

From this it is seen that

$$p_n = \sqrt{k_n^2 - \frac{P}{mA} \left(\frac{n\pi}{l}\right)^2}$$

The above substitutions and simplifications reduce the solution of the differential equations to the solution of the following determinant

$$\begin{vmatrix} p_y^2 - k_n^2 & 0 & c_z k_n^2 \\ 0 & p_z^2 - k_n^2 & -c_y k_n^2 \\ \frac{A}{I_0} c_z k_n^2 & -\frac{A}{I_0} c_y k_n^2 & p_\phi^2 - k_n^2 \end{vmatrix} = 0$$

which gives the following cubic equation

$$(k_n^2)^3 - \frac{I_0}{I_p} [p_y^2 [1 - \frac{Ac_y}{I_0}] + p_z^2 [1 - \frac{Ac_z}{I_0}] + p_\phi^2] (k_n^2)^2 + \frac{I_0}{I_p} [p_z^2 p_y^2 + p_\phi^2 p_y^2 + p_z^2 p_\phi^2] k_n^2 + \frac{I_0}{I_p} p_\phi^2 p_y^2 p_z^2 = 0$$

This equation was solved by hand for k_n^2 and thus for the squares of the natural frequencies p_n^2 as a check upon the general solution obtained in chapter II.

APPENDIX II

The method of solution outlined in chapter II was programmed for a computer in the Fortran IV language. Two programs were used to obtain the natural frequencies of the member for various support conditions with an applied axial load. The first program uses the member constants, value of thrust P , and frequency squared p_n^2 to set up and solve the fifth degree characteristic equation. The output is the square of the roots for the equation. The second program uses the member constants, values of thrust, frequency squared, spring constant β , and roots of the characteristic equation to establish and evaluate the boundary condition determinant. The program writes the value of β , thrust, frequency squared, and of the determinant as output.

To obtain the natural frequency the value of the determinant is plotted versus frequency squared for particular values of thrust and spring constant β . The natural frequency squared is where the line crosses the frequency squared axis.

The two programs used are listed in this appendix. When applying these programs to other problems it should be noted that they are for an unequal leg angle cross section, which means the value of c_w is zero. For a general cross section c_w is not zero and the programs must be changed. The solution for the roots is only for a fifth order equation.

The following is a list of notations used in the programs

HA	A,	area of the cross section square inches
HC	c,	torsion constant in (inches) ⁴

HCI	I_p ,	polar moment of inertia about shear center in (inches) ⁴
DN	m ,	mass density in $\frac{\text{lb.sec}^2}{\text{in}^4}$
DS	λ ,	length of member in inches
EC	e ,	eccentricity, in inches
EI	I_η ,	principal centroidal moment of inertia about η axis in (inches) ⁴
EY	E ,	modulus of elasticity
F	p_n^2 ,	square of circular frequency in (rad./sec) ²
G	G ,	modulus of rigidity in psi
S	P ,	thrust in pounds
SI	I_0 ,	polar moment of inertia about centroidal moment of inertia about the ζ axis in (inches) ⁴
ECY	c_y ,	y coordinate of centroid in inches
ECZ	c_z ,	z coordinate of centroid in inches

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